<u>Continuity and Differentiability - Classwork</u>

Back in our precalculus days, we dabbled in the concept of continuity. We reached a very informal definition of continuity: a curve is continuous if you can draw it without taking your pencil from the paper. This is a good "loose" definition but when one examines it closely, it is filled with holes. For instance, we know that the function $f(x) = x^2$ is a parabola and the parabola is continuous. But can we really draw it in its entirety without taking our pencil from the paper? Try it.

So we need a definition of continuity that is better than the one given above. We will use the following definition for continuity of a function.

A function is continuous at *c* if <u>all three</u> of the following holds:

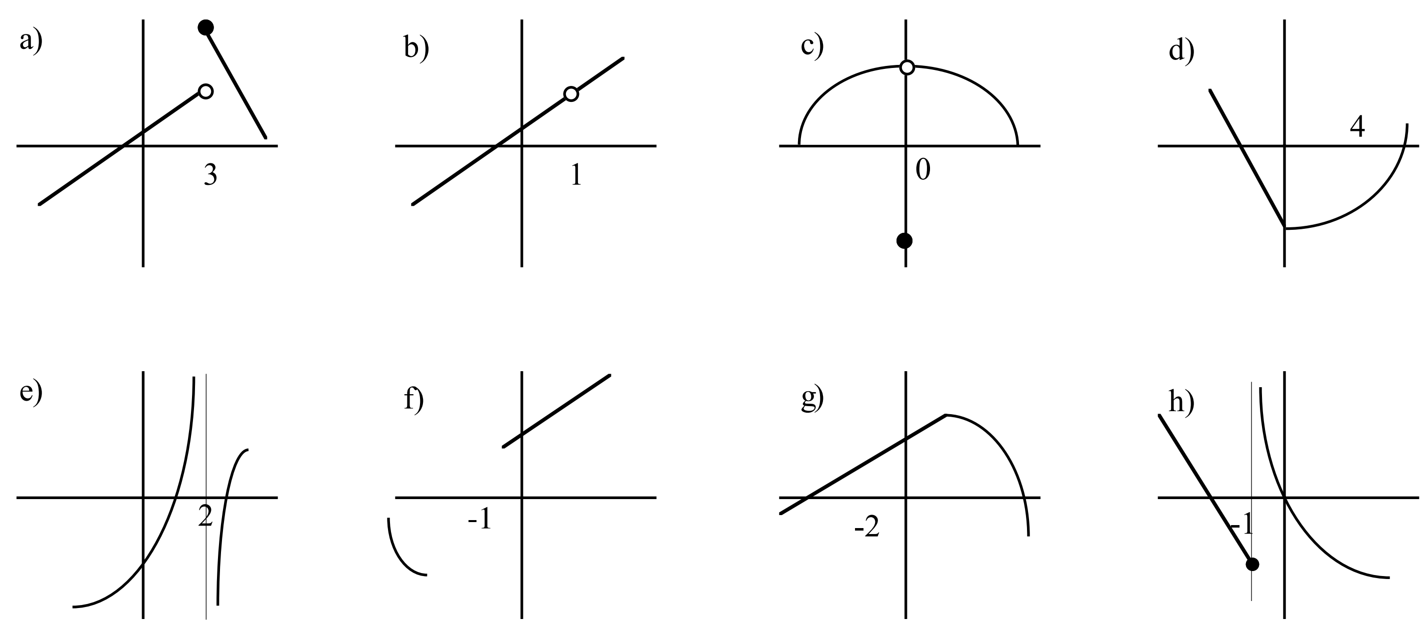
$$\boxed{1.\ \lim_{x\to c} f(x) \text{ exists} \qquad 2.\ f(c) \text{ exists} \qquad 3.\ \lim_{x\to c} f(x) = f(c)}$$

What we are saying is this: for the function to be continuous at some value of *c*:

1) as *x* gets closer and closer to *c* both on the left and right, we approach the same *y*-value.
2) the function is defined at *x* = *c*.
3) the *y*-value limit you got in step 1 is the same as the value you get for the function at *c* in step 2.

If a function is continuous at all values of *x* then we say it is a continuous function. A function can be continuous in certain parts of its domain and discontinuous in others.

Examples: In the following graphs determine if the function $f(x)$ is continuous at the marked value of *c*, and if not, determine which of the 3 rules of continuity the function fails.



Example a) above is an example of a "step" discontinuity. Notice how the function make a step at $x = 3$. Example b) is an example of a removable discontinuity (we usually call it a hole). We will see why it is called removable when we examine it in algebraic form.

When we examine functions in algebraic form, we can make the following conclusions:

a) all polynomials 1. $\lim_{x \to c} f(x)$ exists 2. $f(c)$ exists 3. $\lim_{x \to c} f(x) = f(c)$ are continuous at all values of x.

b) fractions in the form of $y = \dfrac{f(x)}{g(x)}$ are discontinuous wherever $g(x) = 0$.

c) radicals in the form of $y = \sqrt[\text{odd root}]{f(x)}$ are continuous everywhere.

d) radicals in the form of $y = \sqrt[\text{even root}]{f(x)}$ are discontinuous where $f(x) < 0$.

Examples: Find any points of discontinuity of the following functions.

a) $f(x) = -3x^2 - 5x + 1$ b) $f(x) = \dfrac{x-2}{x^2-4}$ c) $f(x) = \sqrt[3]{x^2 + 2x - 1}$ d) $f(x) = \sqrt{x^2 - x - 6}$

In dealing with continuity of a piecewise function, we need to examine the x-value where the rule changes.

Example 3) $f(x) = \begin{cases} x^2 - 3, x \geq 1 \\ 1 - x, x < 1 \end{cases}$

Example 4) $f(x) = \begin{cases} x^2 + 3x - 2, x \geq -2 \\ -x^2, x < -2 \end{cases}$

Example 5) $f(x) = \begin{cases} 3^{-x} - 1, x \geq -1 \\ \dfrac{1}{x+1}, x < -1 \end{cases}$

Example 6) $f(x) = \begin{cases} \dfrac{x-4}{x^2-16}, x \neq 4 \\ \dfrac{1}{3x-4}, x = 4 \end{cases}$

Find the value of the constant k that will make the function continuous. Verify by calculator.

Example 7) $f(x) = \begin{cases} 3x + 2, x \geq 1 \\ 2k - x, x < 1 \end{cases}$

Example 8) $f(x) = \begin{cases} kx^2, x \geq 2 \\ kx - 6, x < 2 \end{cases}$

Example 9) $f(x) = \begin{cases} k^2 - 12x, x \geq 1 \\ kx, x < 1 \end{cases}$

An important concept in calculus involves the concept of **differentiability**. There are several definitions that you need to know:

<u>Definitions</u>

Differentiability at a point: Function $f(x)$ is differentiable at $x = c$ if and only if $f'(c)$ exists. That is, $f'(c)$ is a real number.

Differentiability on an interval: Function $f(x)$ is differentiable on an interval (a,b) if and only if it is diffferentiable for every value of x on the interval (a,b).

Differentiability: Function $f(x)$ is differentiable if and only if it is differentiable at every value of x in its domain.

The concept of differentiability means, in laymans terms, "smooth." A differentiable curve will have no sharp points in it (cusp points) or places where the tangent line to the curve is vertical. Imagine a train traveling on a set of differentiable tracks and you will never get a derailment. Naturally, if a curve is to be differentiable, it must be defined at every point and its limit must exist everywhere. That implies the following:

Differentiability implies Continuity, Continuity does not imply Differentiability.

If a function $f(x)$ is differentiable at $x = c$ then it must be continuous also at $x = c$. $\boxed{D \Rightarrow C}$

However, if a function is continuous at $x = c$, it need not be differentiable at $x = c$. $\boxed{\text{Not! } C \Rightarrow D}$

And, if a function is not continuous, then it can't be differentiable at $x = c$. $\boxed{\text{not } C \Rightarrow \text{ not } D}$

Example: determine whether the following functions are continuous, differentiable, neither, or both at the point.

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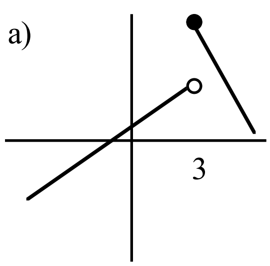

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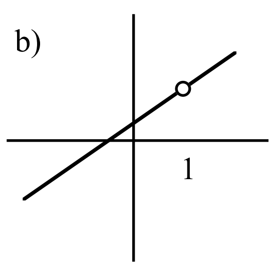

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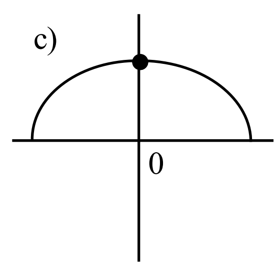

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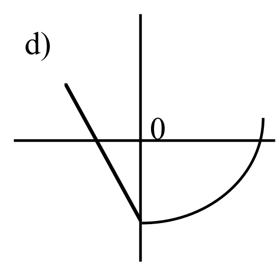

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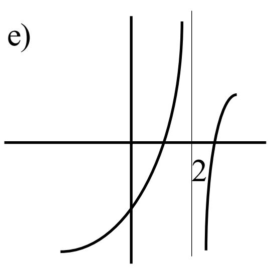

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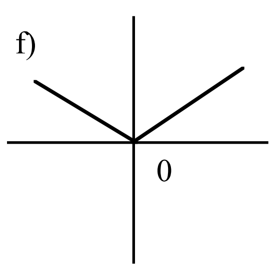

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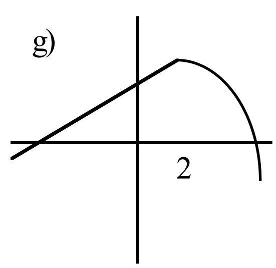

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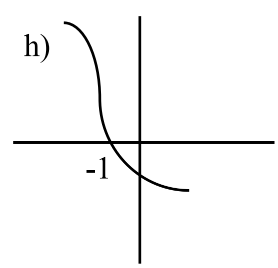


i) $f(x) = x^2 - 6x + 1$

j) $f(x) = \dfrac{x^2 - x - 12}{x + 3}$

k) $f(x) = \sin x$

l) $f(x) = \dfrac{\sin x}{x}$

Example 2) Determine if $f(x)$ is continuous and/or differentiable at the value of the function where the rule changes. Sketch the function.

a) $f(x)=\begin{cases} x^2-6x+10, x\ge 2 \\ 4-x, x<2 \end{cases}$

b) $f(x)=\begin{cases} x^2+x-3, x\ge -1 \\ -x-4, x<-1 \end{cases}$

c) $f(x)=\begin{cases} \sqrt{x+5}, x\ge 4 \\ 4-\sqrt[3]{x-4}, x<4 \end{cases}$

d) $f(x)=\begin{cases} \sin x, x\ge 0 \\ x-3x^2, x<0 \end{cases}$

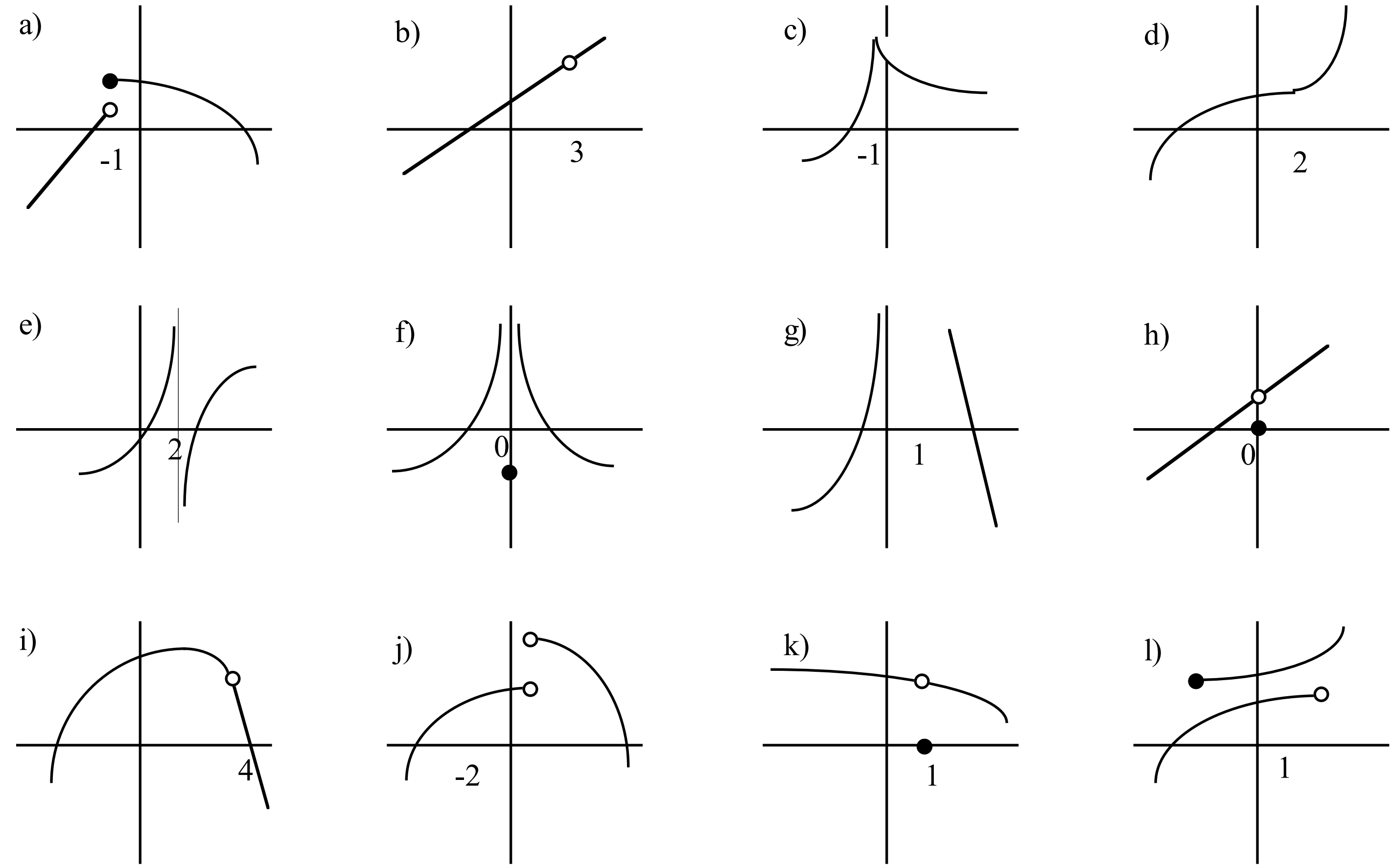
Example 3) Find the values of *a* and *b* that make the function $f(x)$ differentiable.

a) $f(x)=\begin{cases} ax^2+1, x\ge 1 \\ bx-3, x<1 \end{cases}$

b) $f(x)=\begin{cases} ax^3+1, x<2 \\ b(x-3)^2+10, x\ge 2 \end{cases}$

Continuity and Differentiability - Homework

1. In the following graphs determine if the function $f(x)$ is continuous at the marked value of c, and if not, determine for which of the 3 rules of continuity the function fails.



2. Find the value of x where the function is discontinuous.

a. $f(x) = x^3 + 3^x$ b. $f(x) = \dfrac{5}{x^2 - 81}$ c. $f(x) = \dfrac{x^2 + 2x - 24}{x^2 - 36}$ d. $f(x) = \tan x$

3. Find whether the function is continuous at the value where the rule for the function changes.

a. $f(x) = \begin{cases} 8 - x^2, x < 2 \\ 6 - x, x \geq 2 \end{cases}$ b. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 1 + x, x \geq 1 \end{cases}$ c. $f(x) = \begin{cases} 2^x, x < 3 \\ 10 - x, x \geq 3 \end{cases}$

d. $f(x)=\begin{cases}2^{-x}, x<-1\\ x+3, x\geq -1\end{cases}$

e. $f(x)=\begin{cases}\dfrac{1}{x-2}, x<2\\ 3, x=2\\ x+1, x>2\end{cases}$

f. $f(x)=\begin{cases}\dfrac{x^3-x}{x^2-x}, x\neq 0,\ x\neq 1\\ 3, x=0\\ 2, x=1\end{cases}$

4. Find the value of the constant *a* that makes the function continuous.

a. $f(x)=\begin{cases}0.4x+2, x>1\\ 0.3x+a, x\leq 1\end{cases}$

b. $f(x)=\begin{cases}x^2, x>2\\ a-x, x\leq 2\end{cases}$

c. $f(x)=\begin{cases}9-x^2, x>2\\ ax, x\leq 2\end{cases}$

d. $f(x)=\begin{cases}ax+5, x<-1\\ ax^2, x\geq -1\end{cases}$

e. $f(x)=\begin{cases}0.4x+a^2, x<-1\\ ax+1.6, x\geq -1\end{cases}$

f. $f(x)=\begin{cases}a^2-x^2, x<2\\ 1.5ax, x\geq 2\end{cases}$

5. Let *a* and *b* stand for constants and let $f(x)=\begin{cases}b-x, x<1\\ a(x-2)^2, x\geq 1\end{cases}$

a. Find an equation relating *a* and *b* if *f* is to be continuous at *x* = 1.

b. Find *b* if *a* = -1. Graph and show that the function is continuous

c. Find another value of *a*, *b* where *f* is continuous.

6. Graph the function $f(x)=x+4+\dfrac{10^{-25}}{x-2}$ and find what appears to be the limit of $f(x)$ as *x* approaches 2.

7. Sketch a function having the following attributes.

a) has a value of $f(2)$, a limit as *x* approaches 2, but is not continuous at *x* = 2.

b. has a step discontinuity at *x* = 3 where $f(3)=7$

c. $\lim\limits_{x\to 4} f(x)=-2$ but the function is not continuous at *x* = 4.

d. the value of $f(-2)=3$ but there is no limit of $f(x)$ as *x* approaches -2 and no vertical asymptote there.

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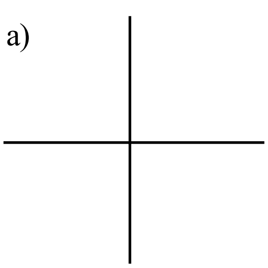

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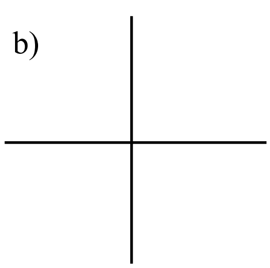

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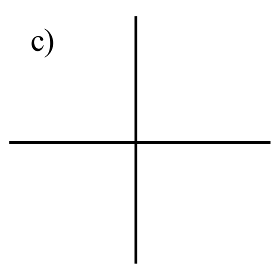

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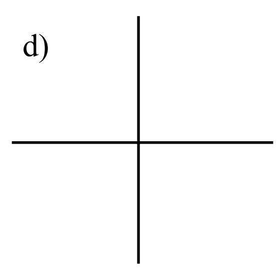
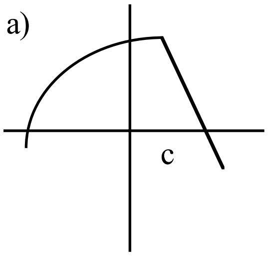


8. For the following, state whether the function is continuous, differentiable, both, or neither at $x = c$

a)



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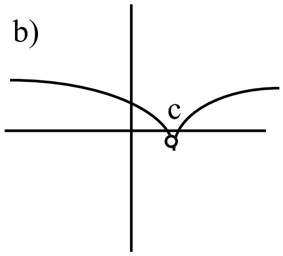

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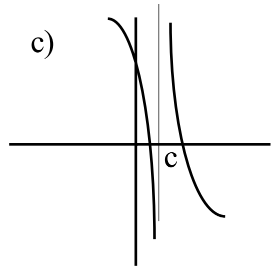

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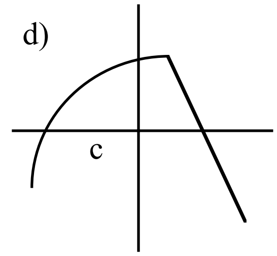

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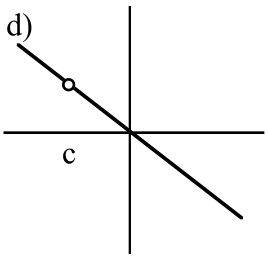

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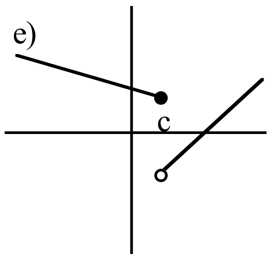

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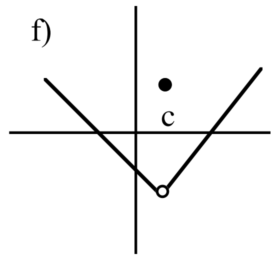

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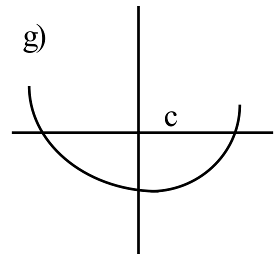


9. Sketch a function having the following attributes.

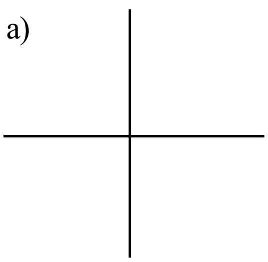
a. is differentiable and continuous at point (2, 4)

b. is continuous at (-3, 1) but not differentiable there

c. has a cusp at the point (-1, 3)

d. is differentiable at (2, -4) but not continuous there.

a)



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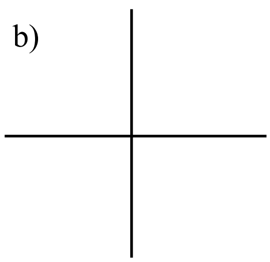

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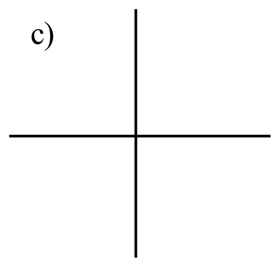

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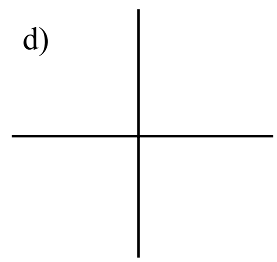

10. For each function, $f(x)$, show work to determine whether the function is continuous or non-continuous, differentiable, or non-differentiable, and sketch the curve. Show work necessary to prove your statements.

a. $f(x) = \begin{cases} x^2, x \geq 0 \\ x, x < 0 \end{cases}$

b. $f(x) = \begin{cases} x^2 + 1, x \geq 0 \\ x^3 + 1, x < 0 \end{cases}$

c. $f(x) = \begin{cases} 4 - x^2, x < 1 \\ 2x + 2, x \geq 1 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

d. $f(x)=\begin{cases} x^2+x-7, x \ge 2 \\ 5x-11, x<2 \end{cases}$

e. $f(x)=\begin{cases} x^4-2x^2, x>1 \\ -1, x \le 1 \end{cases}$

f. $f(x)=\begin{cases} \sqrt{x}-3, x>1 \\ \frac{1}{2}x-\frac{5}{2}, x \le 1 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

g. $f(x)=\begin{cases} \sin(x), x>0 \\ x, x \le 0 \end{cases}$

h. $f(x)=\begin{cases} \cos(x), x \ge 0 \\ 1-x^2, x<0 \end{cases}$

i. $f(x)=\begin{cases} 3+(x+2)^{1/3}, x \ge -2 \\ 3-(x+2)^{2/3}, x<-2 \end{cases}$

cont, diff, both, neither

cont, diff, both, neither

cont, diff, both, neither

11. Find the values of *a* and *b* that make the function $f(x)$ differentiable.

a. $f(x)=\begin{cases} x^3, & x \ge 1 \\ a(x-2)^2+b, & x<1 \end{cases}$

b. $f(x)=\begin{cases} ax^2+10, & x \ge 2 \\ x^2-6x+b, & x<2 \end{cases}$

c. $f(x)=\begin{cases} a/x, & x \ge 1 \\ 12-bx^2, & x<1 \end{cases}$