The AP Calculus Problem Book

$$\int$$

Chuck Garner, Ph.D.

The AP Calculus Problem Book

Publication history:
First edition, 2002
Second edition, 2003
Third edition, 2004
Third edition Revised and Corrected, 2005
Fourth edition, 2006, Edited by Amy Lanchester
Fourth edition Revised and Corrected, 2007
Fourth edition, Corrected, 2008

This book was produced directly from the author's LaTeX files.
Figures were drawn by the author using the TeXdraw package.
TI-Calculator screen-shots produced by a TI-83Plus calculator using a TI-Graph Link.

LaTeX (pronounced "Lay-Tek") is a document typesetting program (not a word processor) that is available free from `www.miktex.org`, which also includes TeXnicCenter, a free and easy-to-use user-interface.

Contents

CHAPTER 1

LIMITS

1.1 Graphs of Functions

Describe the graphs of each of the following functions using only one of the following terms: *line, parabola, cubic, hyperbola, semicircle.*

1. $y = x^3 + 5x^2 - x - 1$

2. $y = \dfrac{1}{x}$

3. $y = 3x + 2$

4. $y = -x^3 + 500x$

5. $y = \sqrt{9 - x^2}$

6. $y = x^2 + 4$

7. $y = \dfrac{-3}{x - 5}$

8. $y = 9 - x^2$

9. $y = -3x^3$

10. $y = 34x - 5^2$

11. $y = 34x^2 - 52$

12. $y = \sqrt{1 - x^2}$

Graph the following functions on your calculator on the window $-3 \le x \le 3$, $-2 \le y \le 2$. Sketch what you see. Choose one of the following to describe what happens to the graph at the origin: A) goes vertical; B) forms a cusp; C) goes horizontal; or D) stops at zero.

13. $y = x^{1/3}$

14. $y = x^{2/3}$

15. $y = x^{4/3}$

16. $y = x^{5/3}$

17. $y = x^{1/4}$

18. $y = x^{5/4}$

19. $y = x^{1/5}$

20. $y = x^{2/5}$

21. Based on the answers from the problems above, find a pattern for the behavior of functions with exponents of the following forms: $x^{\text{even/odd}}$, $x^{\text{odd/odd}}$, $x^{\text{odd/even}}$.

Graph the following functions on your calculator in the standard window and sketch what you see. At what value(s) of x are the functions equal to zero?

22. $y = |x - 1|$

23. $y = |x^2 - 4|$

24. $y = |x^3 - 8|$

25. $y = |4 + x^2|$

26. $y = |x^3| - 8$

27. $y = |x^2 - 4x - 5|$

In the company of friends, writers can discuss their books, economists the state of the economy, lawyers their latest cases, and businessmen their latest acquisitions, but mathematicians cannot discuss their mathematics at all. And the more profound their work, the less understandable it is. —*Alfred Adler*

1.2 The Slippery Slope of Lines

The point-slope form of a line is

$$m(x - x_1) = y - y_1.$$

In the first six problems, find the equation of the line with the given properties.

28. slope: $\frac{2}{3}$; passes through $(2, 1)$

29. slope: $-\frac{1}{4}$; passes through $(0, 6)$

30. passes through $(3, 6)$ and $(2, 7)$

31. passes through $(-6, 1)$ and $(1, 1)$

32. passes through $(5, -4)$ and $(5, 9)$

33. passes through $(10, 3)$ and $(-10, 3)$

34. A line passes through $(1, 2)$ and $(2, 5)$. Another line passes through $(0, 0)$ and $(-4, 3)$. Find the point where the two lines intersect.

35. A line with slope $-\frac{2}{5}$ and passing through $(2, 4)$ is parallel to another line passing through $(-3, 6)$. Find the equations of both lines.

36. A line with slope -3 and passing through $(1, 5)$ is perpendicular to another line passing through $(1, 1)$. Find the equations of both lines.

37. A line passes through $(8, 8)$ and $(-2, 3)$. Another line passes through $(3, -1)$ and $(-3, 0)$. Find the point where the two lines intersect.

38. The function $f(x)$ is a line. If $f(3) = 5$ and $f(4) = 9$, then find the equation of the line $f(x)$.

39. The function $f(x)$ is a line. If $f(0) = 4$ and $f(12) = 5$, then find the equation of the line $f(x)$.

40. The function $f(x)$ is a line. If the slope of $f(x)$ is 3 and $f(2) = 5$, then find $f(7)$.

41. The function $f(x)$ is a line. If the slope of $f(x)$ is $\frac{2}{3}$ and $f(1) = 1$, then find $f(\frac{3}{2})$.

42. If $f(2) = 1$ and $f(b) = 4$, then find the value of b so that the line $f(x)$ has slope 2.

43. Find the equation of the line that has x-intercept at 4 and y-intercept at 1.

44. Find the equation of the line with slope 3 which intersects the semicircle $y = \sqrt{25 - x^2}$ at $x = 4$.

I hope getting the nobel will improve my credit rating, because I really want a credit card. —*John Nash*

1.3 The Power of Algebra

Factor each of the following completely.

45. $y^2 - 18y + 56$

46. $33u^2 - 37u + 10$

47. $c^2 + 9c - 8$

48. $(x-6)^2 - 9$

49. $3(x+9)^2 - 36(x+9) + 81$

50. $63q^3 - 28q$

51. $2\pi r^2 + 2\pi r + hr + h$

52. $x^3 + 8$

53. $8x^2 + 27$

54. $64x^6 - 1$

55. $(x+2)^3 + 125$

56. $x^3 - 2x^2 + 9x - 18$

57. $p^5 - 5p^3 + 8p^2 - 40$

Simplify each of the following expressions.

58. $\dfrac{3(x-4) + 2(x+5)}{6(x-4)}$

59. $\dfrac{1}{x-y} - \dfrac{1}{y-x}$

60. $3x - \dfrac{5x-7}{4}$

61. $\dfrac{\frac{9x^2}{5x^3}}{\frac{3}{x}}$

62. $\dfrac{y}{1 - \frac{1}{y}}$

63. $\dfrac{x}{1 - \frac{1}{y}} + \dfrac{y}{1 - \frac{1}{x}}$

Rationalize each of the following expressions.

64. $\dfrac{-3 + 9\sqrt{7}}{\sqrt{7}}$

65. $\dfrac{3\sqrt{2} + \sqrt{5}}{2\sqrt{10}}$

66. $\dfrac{2x+8}{\sqrt{x+4}}$

67. $\dfrac{2 - \sqrt{3}}{4 + \sqrt{3}}$

68. $\dfrac{x-6}{\sqrt{x-3} + \sqrt{3}}$

69. $\dfrac{9}{\sqrt{2x+3} - \sqrt{2x}}$

70. $\dfrac{5x}{\sqrt{x+5} - \sqrt{5}}$

71. $\dfrac{2\sqrt{5} - 6\sqrt{3}}{4\sqrt{5} + \sqrt{3}}$

72. $\dfrac{x}{\sqrt{x+3} - \sqrt{3}}$

Incubation is the work of the subconscious during the waiting time, which may be several years. Illumination, which can happen in a fraction of a second, is the emergence of the creative idea into the conscious. This almost always occurs when the mind is in a state of relaxation, and engaged lightly with ordinary matters. Illumination implies some mysterious rapport between the subconscious and the conscious, otherwise emergence would not happen. What rings the bell at the right moment? —*John E. Littlewood*

1.4 Functions Behaving Badly

Sketch a graph of each function, then find its domain.

73. $G(x) = \begin{cases} x^2 & x \geq -1 \\ 2x+3 & x < -1 \end{cases}$

74. $A(t) = \begin{cases} |t| & t < 1 \\ -3t+4 & t \geq 1 \end{cases}$

75. $h(x) = x + |x|$

76. $V(r) = \begin{cases} \sqrt{1-r^2} & -1 \leq r \leq 1 \\ \dfrac{1}{r} & r > 1 \end{cases}$

77. $U(x) = \begin{cases} 1/x & x < -1 \\ x & -1 \leq x \leq 1 \\ 1/x & x > 1 \end{cases}$

78. $f(x) = \dfrac{x}{|x|}$

For the following, find a) the domain; b) the y-intercept; and c) all vertical and horizontal asymptotes.

79. $y = \dfrac{x-2}{x}$

80. $y = \dfrac{-1}{(x-1)^2}$

81. $y = \dfrac{x-2}{x-3}$

82. $y = \dfrac{x}{x^2+2x-8}$

83. $y = \dfrac{x^2-2x}{x^2-16}$

84. $y = \dfrac{x^2-4x+3}{x-4}$

Choose the best answer.

85. Which of the following represents the graph of $f(x)$ moved to the left 3 units?

A) $f(x-3)$ B) $f(x)-3$ C) $f(x+3)$ D) $f(x)+3$

86. Which of the following represents the graph of $g(x)$ moved to the right 2 units and down 7 units?

A) $g(x-2)-7$ B) $g(x+2)+7$ C) $g(x+7)-2$ D) $g(x-7)+2$

Factor each of the following.

87. $49p^2 - 144q^2$

88. $15z^2 + 52z + 32$

89. $x^3 - 8$

90. $8x^3 - 27$

91. $27x^3 + y^3$

92. $2w^3 - 10w^2 + w - 5$

He gets up in the morning and immediately starts to do calculus. And in the evening he plays his bongo drums. —*Mrs. Feyman's reasons cited for divorcing her husband, Richard Feyman, Nobel prize-winning physicist*

1.5 Take It to the Limit

Evaluate each limit.

93. $\lim_{x \to -2} (3x^2 - 2x + 1)$

94. $\lim_{x \to 5} 4$

95. $\lim_{x \to -3} (x^3 - 2)$

96. $\lim_{z \to 8} \dfrac{z^2 - 64}{z - 8}$

97. $\lim_{t \to 1/4} \dfrac{4t - 1}{1 - 16t^2}$

98. $\lim_{x \to -2} \dfrac{x^2 + 5x + 6}{x^2 - 4}$

99. $\lim_{x \to 1/3} \dfrac{3x^2 - 7x + 2}{-6x^2 + 5x - 1}$

100. $\lim_{p \to 4} \dfrac{p^3 - 64}{4 - p}$

101. $\lim_{k \to -1} \sqrt[3]{\dfrac{3k - 5}{25k - 2}}$

102. $\lim_{x \to 2} \sqrt{\dfrac{x^2 - 4}{2x^2 + x - 6}}$

103. $\lim_{x \to 0} \dfrac{x}{\sqrt{x + 3} - \sqrt{3}}$

104. $\lim_{y \to 0} \dfrac{\sqrt{3y + 2} - \sqrt{2}}{y}$

105. Let $F(x) = \dfrac{3x - 1}{9x^2 - 1}$. Find $\lim_{x \to 1/3} F(x)$. Is this the same as the value of $F\left(\frac{1}{3}\right)$?

106. Let $G(x) = \dfrac{4x^2 - 3x}{4x - 3}$. Find $\lim_{x \to 3/4} G(x)$. Is this the same as the value of $G\left(\frac{3}{4}\right)$?

107. Let $P(x) = \begin{cases} 3x - 2 & x \neq \frac{1}{3} \\ 4 & x = \frac{1}{3}. \end{cases}$ Find $\lim_{x \to 1/3} P(x)$. Is this the same as the value of $P\left(\frac{1}{3}\right)$?

108. Let $Q(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & x \neq 4 \\ 3 & x = 4. \end{cases}$ Find $\lim_{x \to 4} Q(x)$. Is this the same as the value of $Q(4)$?

Solve each system of equations.

109. $\begin{cases} 2x - 3y = -4 \\ 5x + y = 7 \end{cases}$

110. $\begin{cases} 6x + 15y = 8 \\ 3x - 20y = -7 \end{cases}$

111. If $F(x) = \begin{cases} 2x - 5 & x > \frac{1}{2} \\ 3kx - 1 & x < \frac{1}{2} \end{cases}$ then find the value of k such that $\lim_{x \to 1/2} F(x)$ exists.

1.6 One-Sided Limits

Find the limits, if they exist, and find the indicated value. If a limit does not exist, explain why.

112. Let $f(x) = \begin{cases} 4x-2 & x>1 \\ 2-4x & x \le 1. \end{cases}$

a) $\lim\limits_{x\to 1^+} f(x)$ b) $\lim\limits_{x\to 1^-} f(x)$ c) $\lim\limits_{x\to 1} f(x)$ d) $f(1)$

113. Let $a(x) = \begin{cases} 3-6x & x>1 \\ -1 & x=1 \\ x^2 & x<1. \end{cases}$

a) $\lim\limits_{x\to 1^+} a(x)$ b) $\lim\limits_{x\to 1^-} a(x)$ c) $\lim\limits_{x\to 1} a(x)$ d) $a(1)$

114. Let $h(t) = \begin{cases} 3t-1 & t>2 \\ -5 & t=2 \\ 1+2t & t<2. \end{cases}$

a) $\lim\limits_{t\to 2^+} h(t)$ b) $\lim\limits_{t\to 2^-} h(t)$ c) $\lim\limits_{t\to 2} h(t)$ d) $h(2)$

115. Let $c(x) = \begin{cases} x^2-9 & x<3 \\ 5 & x=3 \\ 9-x^2 & x>3. \end{cases}$

a) $\lim\limits_{x\to 3^+} c(x)$ b) $\lim\limits_{x\to 3^-} c(x)$ c) $\lim\limits_{x\to 3} c(x)$ d) $c(3)$

116. Let $v(t) = |3t-6|$.

a) $\lim\limits_{t\to 2^+} v(t)$ b) $\lim\limits_{t\to 2^-} v(t)$ c) $\lim\limits_{t\to 2} v(t)$ d) $v(2)$

117. Let $y(x) = \dfrac{|3x|}{x}$.

a) $\lim\limits_{x\to 0^+} y(x)$ b) $\lim\limits_{x\to 0^-} y(x)$ c) $\lim\limits_{x\to 0} y(x)$ d) $y(0)$

118. Let $k(z) = |-2z+4|-3$.

a) $\lim\limits_{z\to 2^+} k(z)$ b) $\lim\limits_{z\to 2^-} k(z)$ c) $\lim\limits_{z\to 2} k(z)$ d) $k(2)$

Explain why the following limits do not exist.

119. $\lim\limits_{x\to 0} \dfrac{x}{|x|}$

120. $\lim\limits_{x\to 1} \dfrac{1}{x-1}$

1.7 One-Sided Limits (Again)

In the first nine problems, evaluate each limit.

121. $\displaystyle\lim_{x\to5^+}\frac{x-5}{x^2-25}$

122. $\displaystyle\lim_{x\to2^+}\frac{2-x}{x^2-4}$

123. $\displaystyle\lim_{x\to2}\frac{|x-2|}{x-2}$

124. $\displaystyle\lim_{x\to4^-}\frac{3x}{16-x^2}$

125. $\displaystyle\lim_{x\to0}\frac{x^2-7}{3x^3-2x}$

126. $\displaystyle\lim_{x\to0^-}\left(\frac{3}{x^2}-\frac{2}{x}\right)$

127. $\displaystyle\lim_{x\to2^-}\frac{x+2}{2-x}$

128. $\displaystyle\lim_{x\to4^+}\frac{3x}{x^2-4}$

129. $\displaystyle\lim_{x\to0}\frac{x^2}{\sqrt{3x^2+1}-1}$

Solve each system of equations.

130. $\begin{cases} x-y=-7 \\ \frac{1}{2}x+3y=14 \end{cases}$

131. $\begin{cases} 8x-5y=1 \\ 5x-8y=-1 \end{cases}$

132. If $G(x)=\begin{cases} 3x^2-kx+m & x\ge 1 \\ mx-2k & -1<x<1 \\ -3m+4x^3k & x\le -1 \end{cases}$ then find the values of m and k such that both $\displaystyle\lim_{x\to1}G(x)$ and $\displaystyle\lim_{x\to-1}G(x)$ exist.

For the following, find a) the domain; b) the y-intercept; and c) all vertical and horizontal asymptotes.

133. $y=\dfrac{x^3+3x^2}{x^4-4x^2}$

134. $y=\dfrac{x^5-25x^3}{x^4+2x^3}$

135. $y=\dfrac{x^2+6x+9}{2x}$

Suppose that $\displaystyle\lim_{x\to4}f(x)=5$ and $\displaystyle\lim_{x\to4}g(x)=-2$. Find the following limits.

136. $\displaystyle\lim_{x\to4}f(x)g(x)$

137. $\displaystyle\lim_{x\to4}\left(f(x)+3g(x)\right)$

138. $\displaystyle\lim_{x\to4}\frac{f(x)}{f(x)-g(x)}$

139. $\displaystyle\lim_{x\to4}xf(x)$

140. $\displaystyle\lim_{x\to4}(g(x))^2$

141. $\displaystyle\lim_{x\to4}\frac{g(x)}{f(x)-1}$

How can you shorten the subject? That stern struggle with the multiplication table, for many people not yet ended in victory, how can you make it less? Square root, as obdurate as a hardwood stump in a pasture, nothing but years of effort can extract it. You can't hurry the process. Or pass from arithmetic to algebra; you can't shoulder your way past quadratic equations or ripple through the binomial theorem. Instead, the other way; your feet are impeded in the tangled growth, your pace slackens, you sink and fall somewhere near the binomial theorem with the calculus in sight on the horizon. So died, for each of us, still bravely fighting, our mathematical training; except for a set of people called "mathematicians" – born so, like crooks. —*Stephen Leacock*

1.8 Limits Determined by Graphs

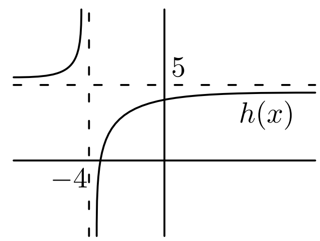
Refer to the graph of $h(x)$ to evaluate the following limits.

142. $\lim_{x \to -4^+} h(x)$

143. $\lim_{x \to -4^-} h(x)$

144. $\lim_{x \to \infty} h(x)$

145. $\lim_{x \to -\infty} h(x)$



Refer to the graph of $g(x)$ to evaluate the following limits.

146. $\lim_{x \to a^+} g(x)$

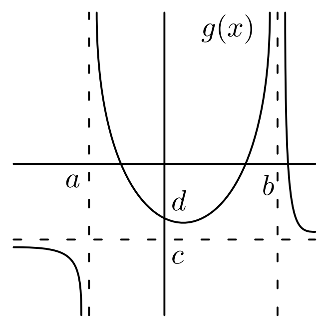
147. $\lim_{x \to a^-} g(x)$

148. $\lim_{x \to 0} g(x)$

149. $\lim_{x \to \infty} g(x)$

150. $\lim_{x \to b^+} g(x)$

151. $\lim_{x \to b^-} g(x)$



Refer to the graph of $f(x)$ to determine which statements are true and which are false. If a statement is false, explain why.

152. $\lim_{x \to -1^+} f(x) = 1$

153. $\lim_{x \to 0^-} f(x) = 0$

154. $\lim_{x \to 0^-} f(x) = 1$

155. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

156. $\lim_{x \to 0} f(x)$ exists

157. $\lim_{x \to 0} f(x) = 0$

158. $\lim_{x \to 0} f(x) = 1$

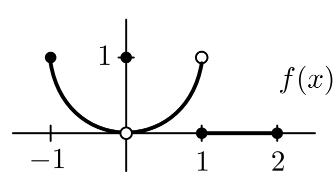
159. $\lim_{x \to 1} f(x) = 1$

160. $\lim_{x \to 1} f(x) = 0$

161. $\lim_{x \to 2^-} f(x) = 2$

162. $\lim_{x \to -1^-} f(x)$ does not exist

163. $\lim_{x \to 2^+} f(x) = 0$



If your experiment needs statistics, you ought to have done a better experiment. —*Ernest Rutherford*

1.9 Limits Determined by Tables

Using your calculator, fill in each of the following tables to five decimal places. Using the information from the table, determine each limit. (For the trigonometric functions, your calculator must be in *radian* mode.)

164. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sqrt{x+3}-\sqrt{3}}{x}$						

165. $\displaystyle\lim_{x\to -3} \frac{\sqrt{1-x}-2}{x+3}$

x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$\frac{\sqrt{1-x}-2}{x+3}$						

166. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin}{x}$						

167. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{1-\cos x}{x}$						

168. $\displaystyle\lim_{x\to 0} (1+x)^{1/x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$(1+x)^{1/x}$						

169. $\displaystyle\lim_{x\to 1} x^{1/(1-x)}$

x	0.9	0.99	0.999	1.001	1.01	1.1
$x^{1/(1-x)}$						

Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house. —*Henri Poincaré*

1.10 The Possibilities Are Limitless...

REFER TO THE GRAPH OF $R(x)$ TO EVALUATE THE FOLLOWING.

170. $\lim_{x\to\infty} R(x)$

171. $\lim_{x\to-\infty} R(x)$

172. $\lim_{x\to a^+} R(x)$

173. $\lim_{x\to a^-} R(x)$

174. $\lim_{x\to a} R(x)$

175. $\lim_{x\to 0} R(x)$

176. $\lim_{x\to b^+} R(x)$

177. $\lim_{x\to b^-} R(x)$

178. $\lim_{x\to b} R(x)$

179. $\lim_{x\to c} R(x)$

180. $\lim_{x\to d} R(x)$

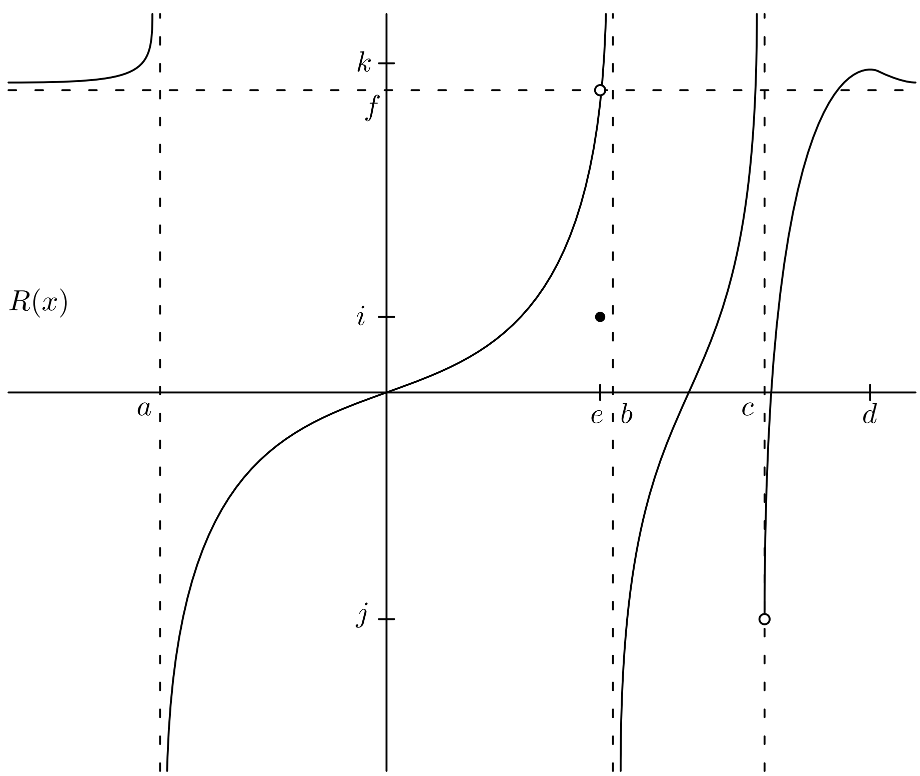
181. $\lim_{x\to e} R(x)$

182. $R(e)$

183. $R(0)$

184. $R(b)$

185. $R(d)$



One of the big misapprehensions about mathematics that we perpetrate in our classrooms is that the teacher always seems to know the answer to any problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics. —*Leon Hankin*

1.11 Average Rates of Change: Episode I

186. Find a formula for the average rate of change of the area of a circle as its radius r changes from 3 to some number x. Then determine the average rate of change of the area of a circle as the radius r changes from

a) 3 to 3.5 b) 3 to 3.2 c) 3 to 3.1 d) 3 to 3.01

187. Find a formula for the average rate of change of the volume of a cube as its side length s changes from 2 to some number x. Then determine the average rate of change of the volume of a cube as the side length s changes from

a) 2 to 3 b) 2 to 2.5 c) 2 to 2.2 d) 2 to 2.1

188. A car is stopped at a traffic light and begins to move forward along a straight road when the light turns green. The distance s, in feet, traveled by a car in t seconds is given by $s(t) = 2t^2$ $(0 \le t \le 30)$. What is the average rate of change of the car from

a) $t = 0$ to $t = 5$ b) $t = 5$ to $t = 10$ c) $t = 0$ to $t = 10$ d) $t = 10$ to $t = 10.1$

In the following six problems, find a formula for the average rate of change of each function from $x = 1$ to some number $x = c$.

189. $f(x) = x^2 + 2x$

190. $f(x) = \sqrt{x}$

191. $f(x) = 2x^2 - 4x$

192. $g(t) = 2t - 6$

193. $p(x) = \dfrac{3}{x}$

194. $F(x) = -2x^3$

1.12 Exponential and Logarithmic Functions

Simplify the following expressions.

195. $e^{\ln x + \ln y}$

196. $\ln(e^{3x})$

197. $\log_4(4^{y+3})$

198. $5^{\log_5(x+2y)}$

199. $\ln(e^{5x+\ln 6})$

200. $e^{3\ln x - 2\ln 5}$

For the following functions, find the domain and the y-intercept.

201. $y = e^{3x-1}\sqrt{x}$

202. $y = x\log_3(5x-2)$

203. $y = e^{3x/(2x-1)}\sqrt[3]{x-7}$

204. $y = \ln(8x^2 - 4)$

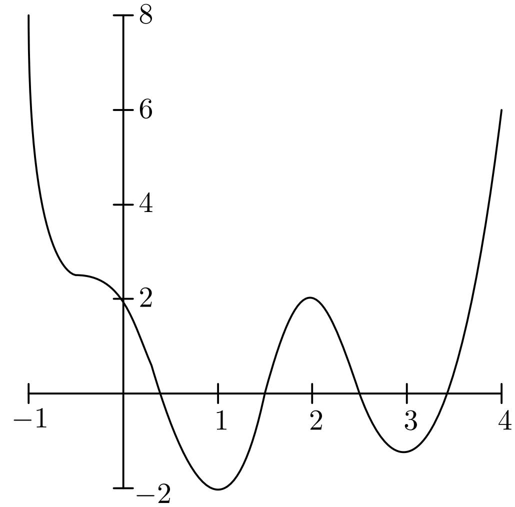
205. $y = e^{5x/(3x-2)}\ln e^x$

206. $y = \ln(x^2 - 8x + 15)$

Ludwig Boltzmann, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on the work, died similarly in 1933. Now it is our turn to study statistical mechanics.
—*David L. Goodstein, in the preface to his book* States of Matter

1.13 Average Rates of Change: Episode II

207. The position $p(t)$ is given by the graph at the right.



a) Find the average velocity of the object between times $t = 1$ and $t = 4$.

b) Find the equation of the secant line of $p(t)$ between times $t = 1$ and $t = 4$.

c) For what times t is the object's velocity positive? For what times is it negative?

208. Suppose $f(1) = 2$ and the average rate of change of f between 1 and 5 is 3. Find $f(5)$.

209. The position $p(t)$, in meters, of an object at time t, in seconds, along a line is given by $p(t) = 3t^2 + 1$.

a) Find the change in position between times $t = 1$ and $t = 3$.

b) Find the average velocity of the object between times $t = 1$ and $t = 4$.

c) Find the average velocity of the object between any time t and another time $t + \Delta t$.

210. Let $f(x) = x^2 + x - 2$.

a) Find the average rate of change of $f(x)$ between times $x = -1$ and $x = 2$.

b) Draw the graph of f and the graph of the secant line through $(-1, -2)$ and $(2, 4)$.

c) Find the slope of the secant line graphed in part b) and then find an equation of this secant line.

d) Find the average rate of change of $f(x)$ between any point x and another point $x + \Delta x$.

Find the average rate of change of each function over the given intervals.

211. $f(x) = x^3 + 1$ over a) $[2, 3]$; b) $[-1, 1]$

212. $R(x) = \sqrt{4x + 1}$ over a) $[0, \frac{3}{4}]$; b) $[0, 2]$

213. $h(t) = \dfrac{1}{\tan t}$ over a) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$; b) $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

214. $g(t) = 2 + \cos t$ over a) $[0, \pi]$; b) $[-\pi, \pi]$

Have lots of ideas and throw away the bad ones. You aren't going to have good ideas unless you have lots of ideas and some sort of principle of selection. —*Linus Pauling*

1.14 Take It To the Limit—One More Time

Evaluate each limit.

215. $\displaystyle\lim_{x\to\infty}\frac{5x-3}{3-2x}$

216. $\displaystyle\lim_{y\to\infty}\frac{4y-3}{3-2y}$

217. $\displaystyle\lim_{x\to\infty}\frac{3x^2+2x+1}{5-2x^2+3x}$

218. $\displaystyle\lim_{x\to\infty}\frac{3x+2}{4x^2-3}$

219. $\displaystyle\lim_{x\to\infty}\frac{4x^2-3}{3x+2}$

220. $\displaystyle\lim_{x\to\infty}\frac{3x^3-1}{4x+3}$

221. $\displaystyle\lim_{x\to\infty}\left(4x+\frac{3}{x^2}\right)$

222. $\displaystyle\lim_{z\to\infty}\frac{\sqrt{z^2+9}}{z+9}$

223. $\displaystyle\lim_{x\to\infty}\frac{3}{x^5}$

224. $\displaystyle\lim_{x\to-2}\frac{5x-1}{x+2}$

225. $\displaystyle\lim_{x\to5}\frac{-4x+3}{x-5}$

226. $\displaystyle\lim_{x\to0}\left(3-\frac{2}{x}\right)$

227. $\displaystyle\lim_{x\to0}\left(3-\frac{2}{x^2}\right)$

228. $\displaystyle\lim_{x\to5}\frac{3x^2}{x^2-25}$

229. $\displaystyle\lim_{x\to0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$

230. $\displaystyle\lim_{x\to-3}\frac{x^2-5x+6}{x^2-9}$

231. $\displaystyle\lim_{x\to-3}(3x+2)$

232. $\displaystyle\lim_{x\to2}(-x^2+x-2)$

233. $\displaystyle\lim_{x\to4}\sqrt[3]{x+4}$

234. $\displaystyle\lim_{x\to2}\frac{1}{x}$

235. $\displaystyle\lim_{x\to3}\frac{\sqrt{x+1}}{x-4}$

236. $\displaystyle\lim_{x\to1}\frac{x^2+x-2}{x^2-1}$

237. $\displaystyle\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

238. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

For the following, a) sketch the graph of f and b) determine at what points c in the domain of f, if any, does $\displaystyle\lim_{x\to c}f(x)$ exist. Justify your answer.

239. $f(x)=\begin{cases}3-x & x<2\\ \dfrac{x}{2}+1 & x>2\end{cases}$

240. $f(x)=\begin{cases}3-x & x<2\\ 2 & x=2\\ \dfrac{x}{2} & x>2\end{cases}$

241. $f(x)=\begin{cases}\dfrac{1}{x-1} & x<1\\ x^3-2x+5 & x\ge2\end{cases}$

242. $f(x)=\begin{cases}1-x^2 & x\ne-1\\ 2 & x=-1\end{cases}$

243. $f(x)=\begin{cases}\sqrt{1-x^2} & 0\le x<1\\ 1 & 1\le x<2\\ 2 & x=2\end{cases}$

244. $f(x)=\begin{cases}x & -1\le x<0 \text{ or } 0<x\le1\\ 1 & x=0\\ 0 & x<-1 \text{ or } x>1\end{cases}$

The discovery in 1846 of the planet Neptune was a dramatic and spectacular achievement of mathematical astronomy. The very existence of this new member of the solar system, and its exact location, were demonstrated with pencil and paper; there was left to observers only the routine task of pointing their telescopes at the spot the mathematicians had marked. —*James R. Newman*

1.15 Solving Equations

Solve each of the following equations.

245. $1 - \dfrac{8}{k^3} = 0$

246. $4p^3 - 4p = 0$

247. $x^3 - 2x^2 - 3x = 0$

248. $3x^2 - 10x - 8 = 0$

249. $|4x^3 - 3| = 0$

250. $|w^2 - 6w| = 9$

251. $\dfrac{3(x-4) - (3x-2)}{(x-4)^2} = 0$

252. $\dfrac{2x-3}{2(x^2-3x)} = 0$

253. $2\ln x = 9$

254. $e^{5x} = 7$

255. $\ln(2x-1) = 0$

256. $e^{3x+7} = 12$

257. $\ln \sqrt[4]{x+1} = \frac{1}{2}$

258. $2^{3x-1} = \frac{1}{2}$

259. $\log_8(x-5) = \frac{2}{3}$

260. $\log \sqrt{z} = \log(z-6)$

261. $2\ln(p+3) - \ln(p+1) = 3\ln 2$

262. $3^{x^2} = 7$

263. $\log_3(3x) = \log_3 x + \log_3(4-x)$

Find all real zeros of the following functions.

264. $y = x^2 - 4$

265. $y = -2x^4 + 5$

266. $y = x^3 - 3$

267. $y = x^3 - 9x$

268. $y = x^4 + 2x^2$

269. $y = x^3 - 4x^2 - 5x$

270. $y = x^3 - 5x^2 - x + 5$

271. $y = x^3 + 3x^2 - 4x - 12$

272. $y = \dfrac{x-2}{x}$

273. $y = \dfrac{-1}{(x-1)^2}$

274. $y = \dfrac{1+x}{1-x}$

275. $y = \dfrac{x^3}{1+x^2}$

276. $y = \dfrac{x^2-2x}{x^2-16}$

277. $y = \dfrac{x^2-4x+3}{x-4}$

278. $y = \dfrac{x^3+3x^2}{x^4-4x^2}$

279. $y = \dfrac{x^5-25x^3}{x^4+2x^3}$

280. $y = x^2 + \dfrac{1}{x}$

281. $y = e^{3x-1}\sqrt{x}$

282. $y = x\log_3(5x-2)$

283. $y = e^{3x/(2x-1)}\sqrt[3]{x-7}$

284. $y = \ln(8x^2-4)$

285. $y = e^{5x/(3x-2)}\ln e^x$

Determine whether the functions in the problems listed are even, odd, or neither.

286. problem 264

287. problem 268

288. problem 272

289. problem 274

290. problem 275

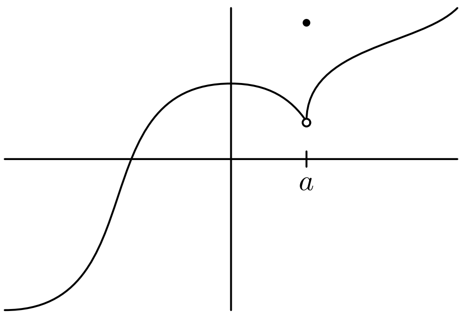
291. problem 280

The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics. —*Johannes Kepler*

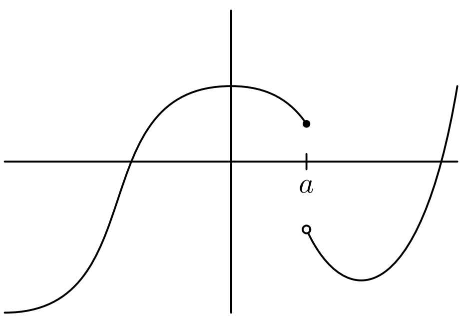
1.16 Continuously Considering Continuity

Examine the graphs of the functions below. Explain why each is discontinuous at $x = a$, and determine the type of discontinuity.

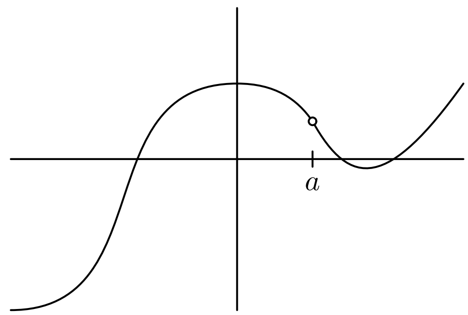
292.



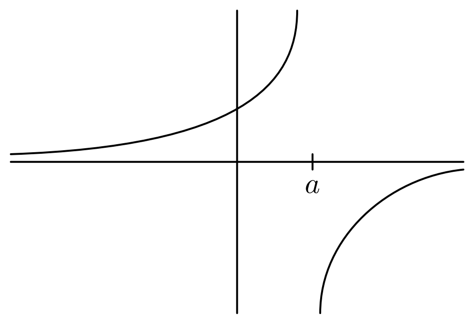
293.



294.



295.



Determine the values of the independent variable for which the function is discontinuous. Justify your answers.

296. $f(x) = \dfrac{x^2 + x - 2}{x - 1}$

297. $d(r) = \dfrac{r^4 - 1}{r^2 - 1}$

298. $A(k) = \dfrac{k^2 - 2}{k^4 - 1}$

299. $q(t) = \dfrac{3}{t + 7}$

300. $m(z) = \begin{cases} \dfrac{z^2 + z - 2}{z - 1} & z \neq 1 \\ 3 & z = 1 \end{cases}$

301. $s(w) = \begin{cases} \dfrac{3}{w + 7} & w \neq -7 \\ 2 & w = -7 \end{cases}$

302. $p(j) = \begin{cases} 4 & j < 0 \\ 0 & j = 0 \\ \sqrt{j} & j > 0 \end{cases}$

303. $b(y) = \begin{cases} y^2 - 9 & y < 3 \\ 5 & y = 3 \\ 9 - y^2 & y > 3 \end{cases}$

Considering how many fools can calculate, it is surprising that it should be thought either a difficult or tedious task for any other fool to learn to master the same tricks. —*Silvanus P. Thompson*

1.17 Have You Reached the Limit?

304. Estimate the value of $\lim_{x\to\infty}(\sqrt{x^2+x+1}-x)$ by graphing or by making a table of values.

305. Estimate the value of $\lim_{x\to\infty}(\sqrt{x^2+x}-\sqrt{x^2-x})$ by graphing or by making a table of values.

306. Consider the function $f(x)=\begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0<x<1 \\ 1 & x=1 \\ -2x+4 & 1<x<2 \\ 0 & 2<x<3. \end{cases}$

a) Graph this function.

b) Does $f(-1)$ exist?

c) Does $\lim_{x\to -1^+} f(x)$ exist?

d) Does $\lim_{x\to -1^+} f(x)=f(-1)$?

e) Is f continuous at $x=-1$?

f) Does $f(1)$ exist?

g) Does $\lim_{x\to 1^+} f(x)$ exist?

h) Does $\lim_{x\to 1^+} f(x)=f(1)$?

i) Is f continuous at $x=1$?

j) Is f defined at $x=2$?

k) Is f continuous at $x=2$?

l) At what values of x is f continuous?

m) What value should be assigned to $f(2)$ to make the function continuous at $x=2$?

n) To what new value of $f(1)$ be changed to remove the discontinuity?

307. Is $F(x)=\dfrac{|x^2-4|x}{x+2}$ continuous everywhere? Why or why not?

308. Is $F(x)=\dfrac{|x^2+4x|(x+2)}{x+4}$ continuous everywhere? Why or why not?

Find the constants a and b such that the function is continuous everywhere.

309. $f(x)=\begin{cases} x^3 & x\le 2 \\ ax^2 & x>2 \end{cases}$

310. $g(x)=\begin{cases} \dfrac{4\sin x}{x} & x<0 \\ a-2x & x\ge 0 \end{cases}$

311. $f(x)=\begin{cases} 2 & x\le -1 \\ ax+b & -1<x<3 \\ -2 & x\ge 3 \end{cases}$

312. $g(x)=\begin{cases} \dfrac{x^2-a^2}{x-a} & x\ne a \\ 8 & x=a \end{cases}$

1.18 Multiple Choice Questions on Limits

313. $\lim\limits_{x\to\infty} \dfrac{3x^4 - 2x + 1}{7x - 8x^5 - 1} =$

A) ∞ B) $-\infty$ C) 0 D) $\frac{3}{7}$ E) $-\frac{3}{8}$

314. $\lim\limits_{x\to 0^-} \dfrac{1}{x} =$

A) ∞ B) $-\infty$ C) 0 D) 1 E) does not exist

315. $\lim\limits_{x\to 1/3} \dfrac{9x^2 - 1}{3x - 1} =$

A) ∞ B) $-\infty$ C) 0 D) 2 E) 3

316. $\lim\limits_{x\to 0} \dfrac{x^3 - 8}{x^2 - 4} =$

A) 4 B) 0 C) 1 D) 3 E) 2

317. In order for the line $y = a$ to be a horizontal asymptote of $h(x)$, which of the following must be true?

A) $\lim\limits_{x\to a^+} h(x) = \infty$

B) $\lim\limits_{x\to a^-} h(x) = -\infty$

C) $\lim\limits_{x\to \infty} h(x) = \infty$

D) $\lim\limits_{x\to -\infty} h(x) = a$

E) $\lim\limits_{x\to -\infty} h(x) = \infty$

318. The function $G(x) = \begin{cases} x - 3 & x > 2 \\ -5 & x = 2 \\ 3x - 7 & x < 2 \end{cases}$ is not continuous at $x = 2$ because

A) $G(2)$ is not defined

B) $\lim\limits_{x\to 2} G(x)$ does not exist

C) $\lim\limits_{x\to 2} G(x) \neq G(2)$

D) $G(2) \neq -5$

E) All of the above

319. $\lim\limits_{x\to 0} \dfrac{3x^2 + 2x}{2x + 1} =$

A) ∞ B) $-\infty$ C) 0 D) 1 E) $\frac{3}{2}$

320. $\displaystyle\lim_{x\to -1/2^-} \frac{2x^2-3x-2}{2x+1} =$

A) ∞ B) $-\infty$ C) 1 D) $\frac{3}{2}$ E) $-\frac{5}{2}$

321. $\displaystyle\lim_{x\to -2} \frac{\sqrt{2x+5}-1}{x+2} =$

A) 1 B) 0 C) ∞ D) $-\infty$ E) does not exist

322. $\displaystyle\lim_{x\to -\infty} \frac{3x^2+2x^3+5}{x^4+7x^2-3} =$

A) 0 B) 2 C) $\frac{3}{7}$ D) ∞ E) $-\infty$

323. $\displaystyle\lim_{x\to 0} \frac{-x^2+4}{x^2-1} =$

A) 1 B) 0 C) -4 D) -1 E) ∞

324. The function $G(x) = \begin{cases} x^2 & x > 2 \\ 4-2x & x < 2 \end{cases}$ is not continuous at $x = 2$ because

A) $G(2)$ does not exist

B) $\displaystyle\lim_{x\to 2} G(x)$ does not exist

C) $\displaystyle\lim_{x\to 2} G(x) = G(2)$

D) All three statements A, B, and C

E) None of the above

325. The domain of the function $f(x) = \sqrt{4-x^2}$ is

A) $x < -2$ or $x > 2$ B) $x \le -2$ or $x \ge 2$ C) $-2 < x < 2$ D) $-2 \le x \le 2$ E) $x \le 2$

326. $\displaystyle\lim_{x\to 5} \frac{x^2-25}{x-5} =$

A) 0 B) 10 C) -10 D) 5 E) does not exist

327. Find k so that $f(x) = \begin{cases} \dfrac{x^2-16}{x-4} & x \ne 4 \\ k & x = 4 \end{cases}$ is continuous for all x.

A) any value B) 0 C) 8 D) 16 E) no value

Insanity means we keep trying the same thing and hope it comes out differently. —*Albert Einstein*

1.19 Sample A.P. Problems on Limits

328. For the function $f(x) = \dfrac{2x-1}{|x|}$, find the following:

a) $\lim\limits_{x\to\infty} f(x)$;

b) $\lim\limits_{x\to-\infty} f(x)$;

c) $\lim\limits_{x\to 0^+} f(x)$;

d) $\lim\limits_{x\to 0^-} f(x)$;

e) All horizontal asymptotes;

f) All vertical asymptotes.

329. Consider the function $h(x) = \dfrac{1}{1-2^{1/x}}$.

a) What is the domain of h?

b) Find all zeros of h.

c) Find all vertical and horizontal asymptotes of h.

d) Find $\lim\limits_{x\to 0^+} h(x)$.

e) Find $\lim\limits_{x\to 0^-} h(x)$.

f) Find $\lim\limits_{x\to 0} h(x)$.

330. Consider the function $g(x) = \dfrac{\sin|x|}{x}$ defined for all real numbers.

a) Is $g(x)$ an even function, an odd function, or neither? Justify your answer.

b) Find the zeros and the domain of g.

c) Find $\lim\limits_{x\to 0} g(x)$.

331. Let $f(x) = \begin{cases} \sqrt{1-x^2} & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & x = 2 \end{cases}$.

a) Draw the graph of f.

b) At what points c in the domain of f does $\lim\limits_{x\to c} f(x)$ exist?

c) At what points does only the left-hand limit exist?

d) At what points does only the right-hand limit exist?

A.P. Calculus Test One
Section One
Multiple-Choice
No Calculators
Time—30 minutes
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where C is the number of correct responses and I is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1. Which of the following is continuous at $x = 0$?

$$\begin{array}{rl} \text{I.} & f(x) = |x| \\ \text{II.} & f(x) = e^x \\ \text{III.} & f(x) = \ln(e^x - 1) \end{array}$$

A) I only

B) II only

C) I and II only

D) II and III only

E) none of these

---

2. The graph of a function f is reflected across the x-axis and then shifted up 2 units. Which of the following describes this transformation on f ?

A) $-f(x)$

B) $f(x) + 2$

C) $-f(x+2)$

D) $-f(x-2)$

E) $-f(x) + 2$

---

3. Which of the following functions is *not* continuous for all real numbers x ?

A) $f(x) = x^{1/3}$

B) $f(x) = \dfrac{2}{(x+1)^4}$

C) $f(x) = |x+1|$

D) $f(x) = \sqrt{1 + e^x}$

E) $f(x) = \dfrac{x-3}{x^2+9}$

4. $\lim_{x\to 1} \dfrac{\ln x}{x}$ is

A) 1

B) 0

C) e

D) $-e$

E) nonexistent

---

5. $\lim_{x\to 0} \left(\dfrac{1}{x} + \dfrac{1}{x^2}\right) =$

A) 0

B) $\frac{1}{2}$

C) 1

D) 2

E) ∞

---

6. $\lim_{x\to\infty} \dfrac{x^3 - 4x + 1}{2x^3 - 5} =$

A) $-\frac{1}{5}$

B) $\frac{1}{2}$

C) $\frac{2}{3}$

D) 1

E) Does not exist

7. For what value of k does $\lim\limits_{x \to 4} \dfrac{x^2 - x + k}{x - 4}$ exist?

A) -12

B) -4

C) 3

D) 7

E) No such value exists.

---

8. $\lim\limits_{x \to 0} \dfrac{\tan x}{x} =$

A) -1

B) $-\frac{1}{2}$

C) 0

D) $\frac{1}{2}$

E) 1

---

9. Suppose f is defined as

$$f(x) = \begin{cases} \dfrac{|x| - 2}{x - 2} & x \neq 2 \\ k & x = 2. \end{cases}$$

Then the value of k for which $f(x)$ is continuous for all real values of x is $k =$

A) -2

B) -1

C) 0

D) 1

E) 2

10. The average rate of change of $f(x) = x^3$ over the interval $[a, b]$ is

A) $3b + 3a$

B) $b^2 + ab + a^2$

C) $\dfrac{b^2 + a^2}{2}$

D) $\dfrac{b^3 - a^3}{2}$

E) $\dfrac{b^4 - a^4}{4(b - a)}$

---

11. The function

$$G(x) = \begin{cases} x - 5 & x > 2 \\ -5 & x = 2 \\ 5x - 13 & x < 2 \end{cases}$$

is not continuous at $x = 2$ because

A) $G(2)$ is not defined.

B) $\lim_{x \to 2} G(x)$ does not exist.

C) $\lim_{x \to 2} G(x) \neq G(2)$.

D) $G(2) \neq -5$.

E) None of the above

---

12. $\lim_{x \to -2} \dfrac{\sqrt{2x + 5} - 1}{x + 2} =$

A) 1

B) 0

C) ∞

D) $-\infty$

E) does not exist

13. The Intermediate Value Theorem states that given a continuous function f defined on the closed interval $[a, b]$ for which 0 is between $f(a)$ and $f(b)$, there exists a point c between a and b such that

A) $c = a - b$

B) $f(a) = f(b)$

C) $f(c) = 0$

D) $f(0) = c$

E) $c = 0$

---

14. The function $t(x) = 2^x - \dfrac{|x-3|}{x-3}$ has

A) a removable discontinuity at $x = 3$.

B) an infinite discontinuity at $x = 3$.

C) a jump discontinuity at $x = 3$.

D) no discontinuities.

E) a removable discontinuity at $x = 0$ and an infinite discontinuity at $x = 3$.

---

15. Find the values of c so that the function

$$h(x) = \begin{cases} c^2 - x^2 & x < 2 \\ x + c & x \geq 2 \end{cases}$$

is continuous everywhere.

A) $-3, -2$

B) $2, 3$

C) $-2, 3$

D) $-3, 2$

E) There are no such values.

A.P. Calculus Test One
Section Two
Free-Response
Calculators Allowed
Time—45 minutes
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $y = x^2$ may not be written as `Y1=X^2`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. Consider the function $f(x) = \dfrac{|x|(x-3)}{9-x^2}$.

a) What is the domain of f ? What are the zeros of f ?

b) Evaluate $\lim_{x\to 3} f(x)$.

c) Determine all vertical and horizontal asymptotes of f.

d) Find all the nonremovable discontinuities of f.

---

2. Consider the function $g(x) = x^x$ with domain $(0, \infty)$.

a) Fill in the following table.

x	0.01	0.1	0.2	0.3	0.4	0.5	1
x^x							

b) What is $\lim_{x\to 1^-} g(x)$? What is $\lim_{x\to 0^+} g(x)$?

c) What do you think the smallest value of $g(x)$ is for values in the interval $(0,1)$? Justify your answer.

d) Find the average rate of change of $g(x)$ from $x = 0.1$ to $x = 0.4$.

---

3. Consider the function $F(x) = (a^{-1} - x^{-1})^{-1}$ where a is a positive real number.

a) What is the domain of F ? What are the zeros of F ?

b) Find all asymptotes of F and discuss any discontinuities of F.

c) Evaluate $\lim_{x\to 0} F(x)$, $\lim_{x\to\infty} F(x)$, and $\lim_{x\to a} F(x)$.

d) For what value of a will $F(6) = 12$?

CHAPTER 2

DERIVATIVES

2.1 Negative and Fractional Exponents

Rewrite each expression with fractional exponents and simplify.

332. $\sqrt[3]{x}\,\sqrt[5]{y^2}$

333. $\sqrt{x+2}\,\sqrt[4]{(x+2)^9}$

334. $x^3\,\sqrt[5]{x^3}$

335. $(x+6)^4\,\sqrt[3]{x+6}$

Rewrite each expression with radicals and simplify.

336. $x^{5/3}$

337. $8(x+2)^{5/2}$

338. $y^{10/3}$

339. $16^{7/4}$

340. $(64x)^{3/2}$

Rewrite and simplify each of the following in two ways: a) with positive exponents only; and b) with no denominators.

341. $\dfrac{x^2y^{-3}}{x^{-4}y^2}$

342. $\dfrac{x^{-2/5}y^{-3/4}}{x^{-3/5}y^{1/4}}$

343. $\dfrac{(x+5)^{-2}(x+7)^3}{(x+7)^4(x+5)^3}$

344. $x^2(x^{-2/3}+x^{-7/3})$

Completely factor each of the following expressions.

345. $2x^{3/5}-4x^{1/5}$

346. $8x^{10/3}+16x^{5/3}+8$

347. $25x^{6/5}-49x^{8/3}$

348. $4x^{-7/3}-6x^{-5/3}+12x^{-1}$

349. $x^3+x^2-x^{-2}-x^{-3}$

350. $(\frac{4}{3}x^{4/3}+2x)(x^{2/3}+4x^{1/3})$

351. $\frac{1}{2}(x^3+3x^2)^{-1/2}(2x+4)$

352. $(x^2+6x+9)^{-1/2}(x+3)^{3/2}$

353. $(x^{-1/3}+x^{-2/3})(x^{1/3}+1)+(x^{2/3}+3x^{1/3}+2)$

354. $\dfrac{\frac{2}{3}(x-2)^{-1/3}x^{4/3}-\frac{4}{3}(x-2)^{2/3}x^{1/3}}{x^{8/3}}$

355. $\dfrac{\frac{1}{2}(x^2+7)^{-1/2}2x\sqrt{x}-\frac{1}{2}x^{-1/2}\sqrt{x^2+7}}{x}$

356. $\dfrac{\frac{1}{2}(x-7)^{-1/2}(x-3)-\sqrt{x-7}}{(x-3)^2}$

2.2 Logically Thinking About Logic

In each of the following problems, you are given a true statement. From the statement, determine which one of the three choices is logically equivalent. (You do not need to know what the words mean in order to determine the correct answer.)

357. If it is raining, then I will go to the mall.

A) If I go to the mall, then it is raining.

B) If it is not raining, then I will not go to the mall.

C) If I do not go to the mall, then it is not raining.

358. If a snark is a grunk, then a quango is a trone.

A) If a quango is a trone, then a snark is a grunk.

B) If a quango is not a trone, then a snark is not a grunk.

C) If a snark is not a grunk, then a quango is not a trone.

359. If a function is linear, then the graph is not a parabola.

A) If the graph is a parabola, then the function is not linear.

B) If the graph is a parabola, then the function is linear.

C) If the function is not linear, then the graph is a parabola.

360. If a function has a vertical asymptote, then it is either rational, logarithmic, or trigonometric.

A) If a function is rational, logarithmic, or trigonometric, then the function has a vertical asymptote.

B) If a function is not rational, logarithmic, and trigonometric, then the function has no vertical asymptote.

C) If a function is neither rational, logarithmic, and trigonometric, then the function has no vertical asymptote.

361. If $f(x)$ is continuous and $f(a) = f(b)$, then there is a number c between a and b so that $f(c)$ is the maximum of $f(x)$.

A) If $f(x)$ is not continuous and $f(a) = f(b)$, then there is not a number c between a and b so that $f(c)$ is the maximum of $f(x)$.

B) If there is a number c between a and b so that $f(c)$ is not the maximum of $f(x)$, then either $f(x)$ is not continuous or $f(a) \neq f(b)$.

C) If there is not a number c between a and b so that $f(c)$ is the maximum of $f(x)$, then $f(x)$ is not continuous or $f(a) \neq f(b)$.

2.3 The Derivative By Definition

For each of the following, use the definition of the derivative to a) find an expression for $f'(x)$ and b) find the value of $f'(a)$ for the given value of a.

362. $f(x) = 2x - 3$; $a = 0$

363. $f(x) = x^2 - x$; $a = 1$

364. $f(x) = \sqrt{1 + 2x}$; $a = 4$

365. $f(x) = \dfrac{1}{x}$; $a = 2$

Differentiate each function. You do not need to use the definition.

366. $g(x) = 3x^2 - 2x + 1$

367. $p(x) = (x - 1)^3$

368. $w(x) = (3x^2 + 4)^2$

369. $J(x) = \dfrac{3x^4 - 2x^3 + 6x}{12x}$

370. $t(x) = \dfrac{5}{2x^3} - \dfrac{3}{5x^4}$

371. $k(x) = (x^{1/3} - 2)(x^{2/3} + 2x^{1/3} + 4)$

372. $y(x) = x^2 - 3x - 5x^{-1} + 7x^{-2}$

373. $G(x) = (3x - 1)(2x + 5)$

374. $S(x) = \sqrt{x} + 17\sqrt[3]{x^2}$

375. $V(x) = \frac{2}{3}\pi x^3 + 10\pi x^2$

Answer each of the following.

376. What is the derivative of any function of the form $y = a$, where a is any constant?

377. What is the derivative of any function of the form $y = mx + b$, where m and b are any constants?

378. What is the derivative of any function of the form $y = x^n$, where n is any constant?

379. If $3x^2 + 6x - 1$ is the derivative of a function, then what could be the original function?

380. Let $y = 7x^2 - 3$. Find y' and $y'(1)$. Find $\dfrac{dy}{dx}$ and $\left.\dfrac{dy}{dx}\right|_{x=2}$.

Determine if each of the following functions is differentiable at $x = 1$; that is, does the derivative exist at $x = 1$?

381. $f(x) = |x - 1|$

382. $f(x) = \sqrt{1 - x^2}$

383. $f(x) = \begin{cases} (x-1)^3 & x \le 1 \\ (x-1)^2 & x > 1 \end{cases}$

384. $f(x) = \begin{cases} x & x \le 1 \\ x^2 & x > 1 \end{cases}$

385. $f(x) = \begin{cases} x^2 & x \le 1 \\ 4x - 2 & x > 1 \end{cases}$

386. $f(x) = \begin{cases} \frac{1}{2}x & x < 1 \\ \sqrt{x} - 1 & x \ge 1 \end{cases}$

A habit of basing convictions upon evidence, and of giving to them only that degree of certainty which the evidence warrants, would, if it became general, cure most of the ills from which the world suffers. —*Bertrand Russell*

2.4 Going Off on a Tangent

For the following five problems, find an equation for the tangent line to the curve at the given x-coordinate.

387. $y = 4 - x^2; \quad x = -1$

388. $y = 2\sqrt{x}; \quad x = 1$

389. $y = x - 2x^2; \quad x = 1$

390. $y = x^{-3}; \quad x = -2$

391. $y = x^3 + 3x; \quad x = 1$

392. At what points does the graph of $y = x^2 + 4x - 1$ have a horizontal tangent?

393. Find an equation for the tangent to the curve $y = \sqrt{x}$ that has slope $\frac{1}{4}$.

394. What is the instantaneous rate of change of the area of a circle when the radius is 3 cm?

395. What is the instantaneous rate of change of the volume of a ball when the radius is 2 cm?

396. Does the graph of $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ have a tangent at the origin? Justify your answer.

397. Consider the curve $y = x^3 - 4x + 1$.

a) Find an equation for the tangent to the curve at the point $(2, 1)$.

b) What is the range of values of the curve's slope?

c) Find equations for the tangents to the curve at the points where the slope of the curve is 8.

Determine which of the following functions are differentiable at $x = 0$.

398. $y = x^{1/3}$

399. $y = x^{2/3}$

400. $y = x^{4/3}$

401. $y = x^{5/3}$

402. $y = x^{1/4}$

403. $y = x^{5/4}$

404. $y = x^{1/5}$

405. $y = x^{2/5}$

406. Based on the answers from the problems above, find a pattern for the differentiability of functions with exponents of the following forms: $x^{\text{even/odd}}$, $x^{\text{odd/odd}}$, $x^{\text{odd/even}}$.

To err is human, but when the eraser wears out ahead of the pencil, you're overdoing it. —*Josh Jenkins*

2.5 Six Derivative Problems

407. Water is flowing into a large spherical tank at a constant rate. Let $V(t)$ be the volume of water in the tank at time t, and $h(t)$ be the height of the water level at time t.

a) Give a physical interpretation of $\dfrac{dV}{dt}$ and $\dfrac{dh}{dt}$.

b) Which of $\dfrac{dV}{dt}$ and $\dfrac{dh}{dt}$ is constant? Explain your answer.

c) Is $\dfrac{dV}{dt}$ positive, negative, or zero when the tank is one quarter full?

d) Is $\dfrac{dh}{dt}$ positive, negative, or zero when the tank is one quarter full?

408. Let $f(x) = 2^x$.

a) Find the average rate of change of f from $x = -1$ to $x = 1$.

b) Find the average rate of change of f from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$.

c) Use your calculator to estimate $f'(0)$, the instantaneous rate of change of f at 0.

d) Sketch the graph of f and use it to explain why the answer to part (b) is a better estimate of $f'(0)$ than the answer to part (a). Can you suggest a generalization of your ideas?

409. The position $p(t)$ of an object at time t is given by $p(t) = 3t^2 + 1$.

a) Find the instantaneous velocity of the object at an arbitrary time t.

b) Find the instantaneous velocity of the object at time $t = -1$.

410. Let $f(x) = x^2 + x - 2$.

a) Use the definition of the derivative to find $f'(x)$.

b) Find an equation of the tangent line to the graph of f at the point $(-1, -2)$.

c) Sketch the graph of f together with the tangent line found in part (b) on the same axes.

411. Find a function $f(x)$ and a point a such that $f'(a)$ does not exist even though $f(a)$ does.

412. There's dust on my guitar! The total amount of dust after t days is given by $g(t)$. I know that $g(30) = 270$ milligrams and $g'(30) = 5$.

a) Estimate $g(32)$.

b) What are the units of $g'(t)$?

Many very learned men have no intelligence. —*Democritus*

Nothing is more terrible than to see ignorance in action. —*Johann Wolfgang von Goethe*

2.6 Trigonometry: a Refresher

Evaluate each of the following expressions. Do not use a calculator.

413. $\tan\frac{\pi}{4}$

414. $(\sin\frac{3\pi}{4})(\cos\frac{5\pi}{4})$

415. $\sec\frac{4\pi}{3}$

416. $\cos(-\frac{\pi}{4})$

417. $\sin(\frac{\pi}{2}-\frac{\pi}{6})$

418. $\sin^2\frac{5\pi}{6}+\tan^2\frac{\pi}{6}$

419. $\arcsin\frac{1}{2}$

420. $\arctan\frac{1}{\sqrt{3}}$

421. $\sin^{-1}(-\frac{\sqrt{3}}{2})$

422. $\tan^{-1}(-\sqrt{3})$

423. $\sin(\arctan 1)$

424. $\tan(\sec^{-1} 2)$

425. $\sin(\arcsin 0.3)$

426. $\arcsin(\sin\pi)$

427. $\arccos\left(\cos(-\frac{\pi}{4})\right)$

428. Which of the following are undefined?

a) $\arccos 1.5$ b) $\operatorname{arcsec} 1.5$ c) $\arctan 1.5$ d) $\operatorname{arcsec} 0.3$ e) $\arcsin 2.4$

Evaluate the following limits. Graph the functions on your calculator if necessary.

429. $\lim_{x\to 1^+}\sin^{-1}x$

430. $\lim_{x\to 1}\sec^{-1}x$

431. $\lim_{x\to 1}\csc^{-1}x$

432. $\lim_{x\to\infty}\arctan x$

433. $\lim_{x\to-\infty}\arctan x$

434. $\lim_{x\to\infty}\operatorname{arcsec} x$

435. We know $\sin x$ is an odd function and $\cos x$ is an even function, but what about these?

a) $\arccos x$ b) $\arcsin x$ c) $\arctan x$ d) $\sec x$ e) $\csc x$

Find exact solutions to each of the following equations over the interval $[0, 2\pi)$.

436. $\cos 3\theta - 1 = 0$

437. $\tan 2x + 1 = 0$

438. $\sin 3\theta + \dfrac{\sqrt{2}}{2} = 0$

439. $2\sin^2\theta - 3\sin\theta + 1 = 0$

440. $2\cos^2\theta + \cos\theta = 0$

441. $\cos x + 2\sec x = -3$

442. Water is draining from a tank. The volume of water in the tank is given by $V(t) = 1000 + (20-t)^3$, where V is in gallons and t is the number of hours since the water began draining. Answer the following questions using correct units.

a) How much water is in the tank initially?

b) How fast is it draining after 10 hours?

c) Will the tank have been completely drained after two days? Why?

Any fool can know. The point is to understand. —*Albert Einstein*

2.7 Continuity and Differentiability

443 (AP). Suppose f is a function for which $\displaystyle\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=0$. Which of the following must be true, might be true, or can never be true?

a) $f'(2)=2$

b) $f(2)=0$

c) $\displaystyle\lim_{x\to 2} f(x)=f(2)$

d) $f(x)$ is continuous at $x=0$.

e) $f(x)$ is continuous at $x=2$.

444 (AP). For some nonzero real number a, define the function f as $f(x)=\begin{cases}\dfrac{x^2-a^2}{x-a} & x\neq a\\ 0 & x=a.\end{cases}$

a) Is f defined at a?

b) Does $\displaystyle\lim_{x\to a} f(x)$ exist? Justify your answer.

c) Is f continuous at a? Justify your answer.

d) Is f differentiable at a? Justify your answer.

445. If $\displaystyle\lim_{x\to a} f(x)=L$, which of the following statements, if any, *must* be true? Justify your answers.

a) f is defined at a.

b) $f(a)=L$.

c) f is continuous at a.

d) f is differentiable at a.

446. Let $f(x)=\begin{cases} ax & x\le 1\\ bx^2+x+1 & x>1.\end{cases}$

a) Find all choices of a and b such that f is continuous at $x=1$.

b) Draw the graph of f when $a=1$ and $b=-1$.

c) Find the values of a and b such that f is differentiable at $x=1$.

d) Draw the graph of f for the values of a and b found in part (c).

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis? —*Jacques Hadamard*

2.8 The RULES: Power Product Quotient Chain

447. Let $f(x) = \begin{cases} 3 - x & x < 1 \\ ax^2 + bx & x \geq 1 \end{cases}$ where a and b are constants.

a) If the function is continuous for all x, what is the relationship between a and b?

b) Find the unique values for a and b that will make f both continuous and differentiable.

448. Suppose that $u(x)$ and $v(x)$ are differentiable functions of x and that

$$u(1) = 2, \qquad u'(1) = 0, \qquad v(1) = 5, \qquad \text{and} \qquad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a) $\dfrac{d}{dx}(uv)$ b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ d) $\dfrac{d}{dx}(7v - 2u)$

449. Graph the function $y = \dfrac{4x}{x^2+1}$ on your calculator in the window $-5 \leq x \leq 5$, $-3 \leq y \leq 3$. (This graph is called *Newton's serpentine.*) Find the tangent lines at the origin and at the point $(1, 2)$.

450. Graph the function $y = \dfrac{8}{x^2+4}$ on your calculator in the window $-5 \leq x \leq 5$, $-3 \leq y \leq 3$. (This graph is called the *witch of Agnesi.*) Find the tangent line at the point $(2, 1)$.

FIND THE DERIVATIVE OF THE GIVEN FUNCTION. EXPRESS YOUR ANSWER IN SIMPLEST FACTORED FORM.

451. $A(z) = (3z - 5)^4$

452. $q(u) = (3u^5 - 2u^3 - 3u - \frac{1}{3})^3$

453. $b(y) = (y^3 - 5)^{-4}$

454. $c(d) = \sqrt[3]{(5d^2 - 1)^5}$

455. $u(p) = \dfrac{3p^2 - 5}{p^3 + 2p - 6}$

456. $V(x) = \dfrac{\sqrt{5x^3}}{5x^3}$

457. $f(x) = 3x^{1/3} - 5x^{-1/3}$

458. $g(z) = \dfrac{1}{\sqrt{36 - z^2}}$

459. $p(t) = (3 - 2t)^{-1/2}$

460. $h(u) = \sqrt{u-1}\sqrt[3]{2u+3}$

461. $f(x) = \dfrac{3x}{x + 5}$

462. $g(y) = \dfrac{4y - 3}{3 - 2y}$

463. $p(x) = \dfrac{x^2 + 10x + 25}{x^2 - 10x + 25}$

464. $m(x) = \dfrac{7x}{1 - 3x}$

465. $f(x) = \dfrac{3}{x^2} - \dfrac{x^2}{3}$

466. $g(x) = \left(\dfrac{4x - 3}{5 - 3x}\right)(2x + 7)$

467. $F(x) = 10x^{27} - 25x^{1/5} + 12x^{-12} + 350$

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction. —*Leo Tolstoy*

2.9 Trigonometric Derivatives

FIND $\dfrac{dy}{dx}$ FOR EACH OF THE FOLLOWING.

468. $y = 3\cos x$

469. $y = \cot x$

470. $y = \tan x - x$

471. $y = x\sin x + \cos x$

472. $y = \sin\left(\frac{3\pi x}{2}\right)$

473. $y = \cos^2 x$

474. $y = \tan^3 x$

475. $y = \sin\sqrt{x}$

476. $y = \cos(3x + 1)$

477. $y = \sin^2(4x)$

478. $y = 2\sin x\cos x$

479. $y = \pi\cot(\pi x)$

480. $y = x^2\tan x$

481. $y = 8\csc 8x$

482. Find all points on the curve $y = \tan x$ over the interval $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ where the tangent line is parallel to the line $y = 2x$.

483. Graph $y = 1 + \sqrt{2}\csc x + \cot x$ on your calculator in the window $0 \le x \le \pi$, $-1 \le y \le 9$. Find the equation of the tangent line at the point $\left(\frac{\pi}{4}, 4\right)$; then find the point on the graph where the graph has a horizontal tangent.

484. Is there a value of b that will make $g(x) = \begin{cases} x + b & x < 0 \\ \cos x & x \ge 0 \end{cases}$ continuous at $x = 0$? Differentiable at $x = 0$? Justify your answers.

485. Find the 1000th derivative of $\cos x$.

486. Find the tangent to the curve $y = 2\tan\left(\frac{\pi x}{4}\right)$ at $x = 1$.

FIND y'' FOR EACH OF THE FOLLOWING.

487. $y = \csc\theta$

488. $y = \sec\theta$

489. $y = 2 - 2\sin\theta$

490. $y = \sin\theta + \cos\theta$

Neither in the subjective nor in the objective world can we find a criterion for the reality of the number concept, because the first contains no such concept, and the second contains nothing that is free from the concept. How then can we arrive at a criterion? Not by evidence, for the dice of evidence are loaded. Not by logic, for logic has no existence independent of mathematics: it is only one phase of this multiplied necessity that we call mathematics. How then shall mathematical concepts be judged? They shall not be judged. Mathematics is the supreme arbiter. From its decisions there is no appeal. We cannot change the rules of the game, we cannot ascertain whether the game is fair. We can only study the player at his game; not, however, with the detached attitude of a bystander, for we are watching our own minds at play. —*Dantzig*

2.10 Tangents, Normals, and Continuity (Revisited)

491. Find the equation of the tangent line to the curve $y = \sqrt{x^2 - 3}$ at the point $(2, 1)$.

492. Find the equation of the normal line to the curve $y = (3x - 1)^2(x - 1)^3$ at $x = 0$.

493. Find the equation of the tangent line to the curve $y = \sqrt{3x - 1}$ that is perpendicular to the line $3y + 2x = 3$.

494. Find the equation of the normal line to the curve $y = x\sqrt{25 + x^2}$ at $x = 0$.

495. Find the equation of the tangent line to the curve $y = \dfrac{2 - x}{5 + x}$ at $x = 1$.

496. Find the equation of the normal line to the curve $y = \dfrac{5}{(5 - 2x)^2}$ at $x = 0$.

497. Find the equation of the tangent line to the curve $y = 3x^4 - 2x + 1$ that is parallel to the line $y - 10x - 3 = 0$.

498. The point $P(3, -2)$ is not on the graph of $y = x^2 - 7$. Find the equation of each line tangent to $y = x^2 - 7$ that passes through P.

For the following six problems, determine if f is differentiable at $x = a$.

499. $f(x) = |x + 5|$; $a = -5$

500. $f(x) = \begin{cases} x + 3 & x \leq -2 \\ -x - 1 & x > -2 \end{cases} \quad a = -2$

501. $f(x) = \begin{cases} 2 & x < 0 \\ x - 4 & x \geq 0 \end{cases} \quad a = 0$

502. $f(x) = \begin{cases} -2x^2 & x < 0 \\ 2x^2 & x \geq 0 \end{cases} \quad a = 0$

503. $f(x) = \begin{cases} x^2 - 5 & x < 3 \\ 3x - 5 & x \geq 3 \end{cases} \quad a = 3$

504. $f(x) = \begin{cases} \sqrt{2 - x} & x < 2 \\ (2 - x)^2 & x \geq 2 \end{cases} \quad a = 2$

505. Suppose that functions f and g and their first derivatives have the following values at $x = -1$ and at $x = 0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	0	-1	2	1
0	-1	-3	-2	4

Evaluate the first derivatives of the following combinations of f and g at the given value of x.

a) $3f(x) - g(x), \quad x = -1$

b) $[f(x)]^3[g(x)]^3, \quad x = 0$

c) $g(f(x)), \quad x = -1$

d) $f(g(x)), \quad x = -1$

e) $\dfrac{f(x)}{g(x) + 2}, \quad x = 0$

f) $g(x + f(x)), \quad x = 0$

Number rules the universe. —*Pythagoras*

2.11 Implicit Differentiation

Find $\dfrac{dy}{dx}$ for each of the following.

506. $x^2 - y^2 = 5$

507. $1 - xy = x - y$

508. $y^2 = x^3$

509. $x = \tan y$

510. $x^3 - xy + y^3 = 1$

511. $9x^2 + 25y^2 = 225$

512. Find the equation of both the tangent and normal lines to the curve $x^2 \cos^2 y - \sin y = 0$ at the point $(0, \pi)$.

513. Find the equation of both the tangent and normal lines to the curve $y^2(2 - x) = x^3$ at the point $(1, 1)$.

Find $\dfrac{d^2y}{dx^2}$ in terms of x and y for the following three problems.

514. $xy + y^2 = 1$

515. $y^2 = x^2 + 2x$

516. $x^2 + xy = 5$

517. Find the equation of the tangent line to the curve $(x^2 + y^2)^2 = 4x^2y$ at the point $(1, 1)$.

518. Consider the curve defined by $x^3 + y^3 - 9xy = 0$.

a) Find the equation of the tangent lines at the points $(4, 2)$ and $(2, 4)$.

b) At what points does the curve have a horizontal tangent?

c) Find the coordinates of the point where the curve has a vertical tangent.

519. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

520. The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at the point $(1, 1)$ intersects the curve at what other point?

521 (AP, 2000AB). Consider the curve given by $xy^2 - x^3y = 6$.

a) Find $\dfrac{dy}{dx}$.

b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

c) Find the x-coordinate of each point on the curve where the tangent is vertical.

My faults are infinite, but modesty prevents me from mentioning them all. —*Stanislaw Ulam*

2.12 The Return of Geometry

522. Find the area and circumference of a circle of radius 7.

523. Find the volume of a cylinder with radius 8 and height 10.

524. Find the volume and surface area of a sphere of radius 9.

525. Find the volume and surface area of a cube of side length 6.

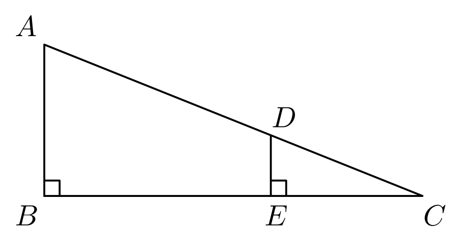
526. Find the volume and surface area of a box with dimensions 3, 4, and 5.

527. What is the hypotenuse of a right triangle with legs 5 and 12?

528. The area of an isosceles right triangle is 8. What is the length of its hypotenuse?

529. A cylinder is constructed so that its height is exactly 4 times its radius. If the volume of the cylinder is 500π, then what is its radius?

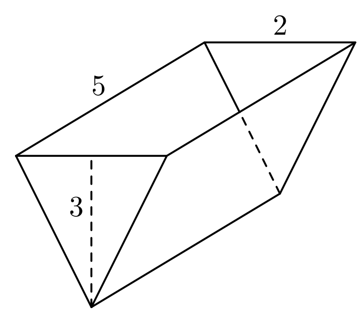
530. In the figure to the right, $DE = 2$, $EC = 5$, and $AB = 5$. Find the lengths of $\overline{AC}$ and $\overline{BC}$.



531. What is the area of an equilateral triangle if its side lengths are 8?

532. What is the area of a semicircle of radius 10?

533. The trough shown in the figure at the right is 5 feet long and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Find the volume of water in the trough when the trough is full.



534. A cone is constructed so that its height is exactly 4 times its radius. If the volume of the cone is 324π, then what is its radius?

535. A 12-foot ladder is leaning against a wall so that it makes a $60°$ angle with the ground. How high up the wall does the ladder touch the wall?

536. An equilateral triangle has an area of $4\sqrt{3}$. What is the height of this equilateral triangle?

All the pictures which science now draws of nature and which alone seem capable of according with observational fact are mathematical pictures. —*Sir James Hopwood Jeans*

2.13 Meet the Rates (They're Related)

SOLVE THE FOLLOWING PROBLEMS, ASSUMING THAT ALL VARIABLES ARE FUNCTIONS OF THE VARIABLE t.

537. If $xy = -3$ and $\dfrac{dx}{dt} = 1$, then find $\dfrac{dy}{dt}$ when $x = 6$.

538. If $x^2 - y^2 = 39$ and $\dfrac{dx}{dt} = 2$, then find $\dfrac{dy}{dt}$ when $y = 5$.

539. If $\dfrac{y}{z} = 13$ and $\dfrac{dz}{dt} = -2$, then find $\dfrac{dy}{dt}$ when $y = 26$.

SOLVE EACH OF THE FOLLOWING PROBLEMS.

540. The volume of a cube is decreasing at the rate of 10 m^3/hr. How fast is the total surface area decreasing when the surface area is 54 m^2?

541. The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of a) the area; b) the perimeter; and c) the length of the diagonal of the rectangle. Which of these quantities are decreasing and which are increasing?

542. Rachael is blowing up a balloon so that the diameter increases at the rate of 10 cm/sec. At what rate must she blow air into the balloon when the diameter measures 4 cm?

543. Assume Clark and Lana leave Smallville Stadium from the same point at the same time. If Clark runs south at 4 mph and Lana runs west at 3 mph, how fast will the distance between Clark and Lana be changing at 10 hours?

544. Suppose Aaron is pumping water into a tank (in the shape of an inverted right circular cone) at a rate of 1600 ft^3/min. If the altitude is 10 ft and the radius of the base is 5 ft, find the rate at which the radius is changing when the height of the water is 7 ft.

545. LuthorCorp Industries hires Professor Patel to calculate the revenue and cost of their best-selling pesticide. Professor Patel finds that the revenue is $R(x) = 700x - \dfrac{x^2}{5000}$ and the cost is $C(x) = 300 + 4x$, where x is the number of gallons of pesticide produced each week. If the rate of production is increasing by 50 gallons per week, and the current production is 300 gallons per week, find the rate of change in a) the revenue R; b) the cost C; and c) the profit $P = R - C$.

546. The area of an equilateral triangle is increasing at the rate of 5 m^2/hr. Find the rate at which the height is changing when the area is $\dfrac{64}{\sqrt{3}}$ m^2.

547. The talented Ed Wynwyte is flying a kite at a constant height of 400 meters. The kite is moving horizontally at a rate of 30 m/sec. How fast must he unwind the string when the kite is 500 m away from him?

Physics is much too hard for physicists. —*David Hilbert*

2.14 Rates Related to the Previous Page

548. A ladder 15 feet tall leans against a vertical wall of a home. If the bottom of the ladder is pulled away horizontally from the house at 4 ft/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 9 feet from the wall?

549. A cone (vertex down) with height 10 inches and radius 2 inches is being filled with water at a constant rate of 2 in^3/sec. How fast is the surface of the water rising when the depth of the water is 6 inches?

550. A particle is moving along the graph of $y = \sqrt{x}$. At what point on the curve are the x-coordinate and y-coordinate of the particle changing at the same rate?

551. A streetlight is 15 feet above the sidewalk. Jonathan, who is 7 feet tall, walks away from the light at the rate of 5 feet per second.

a) Determine a function relating the length of Jonathan's shadow to his distance from the base of the streetlight.

b) Determine the rate at which Jonathan's shadow is lengthening at the moment that he is 20 feet from the base of the light.

552. A spherical balloon is inflated with helium at the rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

553. On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 feet long. At the moment in question, the angle θ the sun's rays make with the ground is increasing at the rate of $\frac{3\pi}{2000}$ radian/min. At what rate is the shadow decreasing? (Express your answer in inches per minute.)

554 (AP, 1970AB). A right circular cone and a hemisphere have the same base, and the cone is inscribed in the hemisphere. The figure is expanding in such a way that the combined surface area of the hemisphere and its base is increasing at a constant rate of 18 square inches per second. At what rate is the volume of the cone changing at the instant when the radius of the common base is 4 inches?

555 (AP, 1976AB). Consider the hyperbola $3x^2 - y^2 = 23$.

a) A point moves on the hyperbola so that its y-coordinate is increasing at a constant rate of 4 units per second. How fast is the x-coordinate changing when $x = 4$?

b) For what value of k will the line $2x + 9y + k = 0$ be normal to the hyperbola?

In the mathematics I can report no deficience, except that it be that men do not sufficiently understand the excellent use of the pure mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect that it maketh a quick eye and a body ready to put itself into all postures; so in the mathematics, that use which is collateral and intervenient is no less worthy than that which is princpial and intended. —*Roger Bacon*

2.15 Excitement with Derivatives!

Find y' for each of the following.

556. $y = e^{2x}$

557. $y = e^{-3x/2}$

558. $y = x^2 e^x$

559. $y = 5e^{2-x}$

560. $y = 8^{2x}$

561. $y = 3^{x^2}$

562. $y = 2^{\sin x}$

563. $y = 9^{-x}$

564. $y = \dfrac{e^{5x}}{x^2}$

565. $y = \ln(x^2)$

566. $y = \ln(2 - x^2)$

567. $y = \ln(5x + 1)$

568. $y = \ln(\sin x)$

569. $y = (\ln x)^2$

570. $y = \log_3(1 + x)$

571. $y = \log_9 \sqrt{x}$

572. $y = x \ln x - x$

573. $y = \dfrac{\ln x}{x^2}$

Find the derivative of each function in simplest factored form.

574. $g(x) = x^3 e^{2x}$

575. $Z(x) = 4e^{4x^2+5}$

576. $q(x) = \ln(e^x + 1)$

577. $f(x) = \dfrac{e^x - 1}{e^x + 1}$

578. $k(x) = \log_3(x^2 + e^x)$

579. $R(x) = \dfrac{2^x - 1}{5^x}$

580. $D(x) = \ln(\ln x)$

581. $A(x) = \ln(x^2 + x + 1)^2$

582. $q(x) = \ln \sqrt[5]{3x - 2}$

583. $A(x) = \dfrac{\ln x}{x - 2}$

584. $B(x) = \dfrac{x - 2}{\ln x}$

585. $M(x) = e^{-2x^3}$

586. $J(x) = \dfrac{e^x}{x^3}$

587. $F(x) = x^2 e^{-4 \ln x}$

588. $f(x) = 10^{3x^2 - 6x}$

589. $g(x) = 3^{2x} 2^{3x^2}$

Use implicit differentiation to find $\dfrac{dy}{dx}$.

590. $2x - 3y + \ln(xy) = 4$

591. $4x = \ln(x + 3y - 4) + 5$

592. $\ln e^x - \ln y = e^y$

593. $y = 4 \sin(x - 3y)$

594. $2x = 3 \sin y - 2y$

595. $\cos(x - 2y) = 3y$

Find $\dfrac{dy}{dx}$ in simplest factored form.

596. $y = 3x \csc 2x$

597. $y = \dfrac{\cot 5x}{3x^2}$

598. $y = \sqrt{\cot 5x}$

599. $y = 3 \sin 8x \cos 8x$

600. $y = \dfrac{\ln x}{\sin x}$

601. $y = \cos^2 3x - \sin^2 3x$

602. $y = e^{\sin x}$

603. $y = 3^{\cos x}$

604. $y = \log_3(\sin 2x)$

605. $y = xe^{\ln 3x}$

606. $y = e^{3x} \tan x$

607. $y = e^{1/x^2}$

608. $y = e^{x^2/4}$

609. $y = \ln(\sec x + \tan x)$

610. $y = xe^{\tan x}$

Mathematics is queen of the sciences. —*Eric Temple Bell*

2.16 Derivatives of Inverses

Find the inverse f^{-1} of the following functions f.

611. $f(x) = \sqrt[3]{x}$

612. $f(x) = \sqrt{x-1}$

613. $f(x) = \dfrac{x+2}{3}$

614. $f(x) = \dfrac{1}{x}$

615. $f(x) = e^{2x}$

616. $f(x) = \ln(x-3)$

617. $f(x) = 5^{2x-1}$

618. $f(x) = \log_2 x$

619. $f(x) = \dfrac{2}{x+5}$

Find the derivative of the inverse of F at the point $x = d$.

620. $F(x) = x^3 - 4;\;\; d = 23$

621. $F(x) = \sqrt{2x-5};\;\; d = 1$

622. $F(x) = x^2 - 9, x \geq 0;\;\; d = 7$

623. $F(x) = 4x^5 + 3x^3;\;\; d = 7$

624. $F(x) = 2x^2 + 10x + 13, x > -\frac{5}{2};\;\; d = 1$

625. $F(x) = \sin x;\;\; d = \frac{1}{2}$

626. $F(x) = \tan x;\;\; d = 1$

627. $F(x) = 17x^3;\;\; d = 17$

628. $F(x) = x + \sin x;\;\; d = 0$

629. $F(x) = \sqrt[3]{x^2-4};\;\; d = \sqrt[3]{5}$

Find y' for each of the following.

630. $y = \sec^{-1}(5x)$

631. $y = \cos^{-1}(2x-3)$

632. $y = \arctan(2x-3)$

633. $y = \operatorname{arcsec}(3x^2)$

634. $y = \tan^{-1}\left(\dfrac{3}{x}\right)$

635. $y = \arccos\left(\dfrac{1}{x}\right)$

636. $y = 2\sin^{-1}\sqrt{1-2x^2}$

637. $y = \arcsin(1-x)$

638. Find an equation for the line tangent to the graph of $y = e^x$ and that goes through the origin.

An expert is someone who knows some of the worst mistakes that can be made in his subject, and how to avoid them. —*Werner Heisenberg*

2.17 Dérivé, Derivado, Ableitung, Derivative

639. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	-3	$\frac{1}{2}$
1	3	5	$\frac{1}{2}$	-4

Find the first derivatives of the following combinations at the given value of x.

a) $6f(x) - g(x)$ at $x = 1$

b) $f(x)g^2(x)$ at $x = 0$

c) $\dfrac{f(x)}{g(x)+1}$ at $x = 1$

d) $f(g(x))$ at $x = 0$

e) $g(f(x))$ at $x = 0$

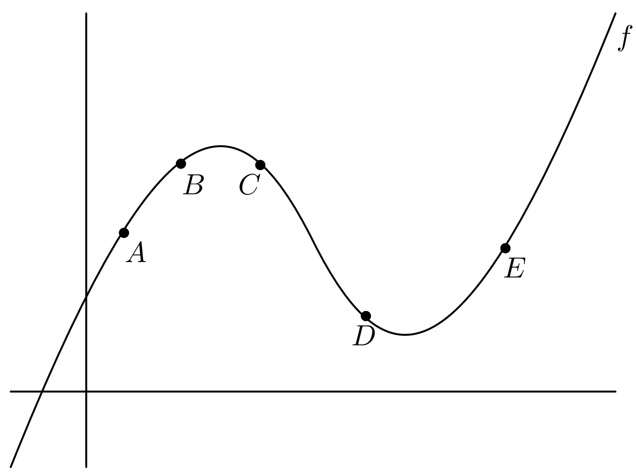
f) $(x + f(x))^{3/2}$ at $x = 1$

g) $f(x + g(x))$ at $x = 0$

640. If $x^2 - y^2 = 1$, find $\dfrac{d^2y}{dx^2}$ at the point $(2, \sqrt{3})$.

641. For what values of a and b will $f(x) = \begin{cases} ax & x < 2 \\ ax^2 - bx + 3 & x \geq 2 \end{cases}$ be differentiable for all values of x?

642. Use the graph of f to answer the following.



a) Between which two consecutive points is the average rate of change of the function greatest? Least?

b) Is the average rate of change of the function between A and B greater than or less than the instantaneous rate of change at B?

c) Sketch a tangent line to the graph between the points D and E such that the slope of the tangent is the same as the average rate of the change of the function between D and E.

d) Give a set of two points for which the average rate of change of the function is approximately equal to another set of two points.

643. The displacement from equilibrium of an object in harmonic motion on the end of a spring is $y = \frac{1}{3}\cos(12t) - \frac{1}{4}\sin(12t)$ where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \frac{\pi}{8}$.

644. The yield Y, in millions of cubic feet per acre, for a stand of timber at age t is $Y = 6.7e^{-48.1/t}$ where t is measured in years.

a) Find the limiting volume of wood per acre as t approaches infinity.

b) Find the rate at which the yield is changing when $t = 20$ years and $t = 60$ years.

645. Find expressions for the velocity and acceleration of a particle whose position is given by $x(t) = \sqrt{t} + \sin t$.

646. The position of a particle is given by $x(t) = t^3 - 9t^2 + 6t - 3$. Find the value of the position and velocity of the particle at the time when the acceleration is zero.

647. A ball thrown follows a path described by $y = x - 0.02x^2$.

a) Sketch a graph of the path.

b) Find the total horizontal distance the ball was thrown.

c) At what x-value does the ball reach its maximum height?

d) Find an equation that gives the instantaneous rate of change of the height of the ball with respect to the horizontal change. Evaluate this equation at $x = 0$, 10, 25, 30, and 50.

e) What is the instantaneous rate of change of the height when the ball reaches its maximum height?

648. A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \arctan t$.

a) Prove that the particle is always moving to the right.

b) Prove that the particle is always decelerating.

c) What is the limiting position of the particle as t approaches infinity?

649. The position at time $t \geq 0$ of a particle moving along a coordinate line is $x = 10\cos(t + \frac{\pi}{4})$.

a) What is the particle's starting position?

b) What are the points farthest to the left and right of the origin reached by the particle?

c) Find the particle's velocity and acceleration at the points in part (b).

d) When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

No pain, no gain. —*Arnold Schwarzennegger*

2.18 Sample A.P. Problems on Derivatives

650. Let $f(x) = \begin{cases} x^2 & x \le 1 \\ 2x & x > 1. \end{cases}$

a) Find $f'(x)$ for $x < 1$.

b) Find $f'(x)$ for $x > 1$.

c) Find $\lim\limits_{x\to 1^-} f'(x)$.

d) Find $\lim\limits_{x\to 1^+} f'(x)$.

e) Does $f'(1)$ exist? Explain.

651. Let f be the function with derivative $f'(x) = \sin(x^2)$ and $f(0) = -1$.

a) Find the tangent line to f at $x = 0$.

b) Use your answer to part (a) to approximate the value of f at $x = 0.1$.

c) Is the actual value of f at $x = 0.1$ greater than or less than the approximation from part (b)? Justify your answer.

652 (1987AB). Let $f(x) = \sqrt{1 - \sin x}$.

a) What is the domain of f?

b) Find $f'(x)$.

c) What is the domain of f'?

d) Write an equation for the line tangent to the graph of f at $x = 0$.

653 (1994AB). Consider the curve defined by $x^2 + xy + y^2 = 27$.

a) Write an expression for the slope of the curve at any point (x, y).

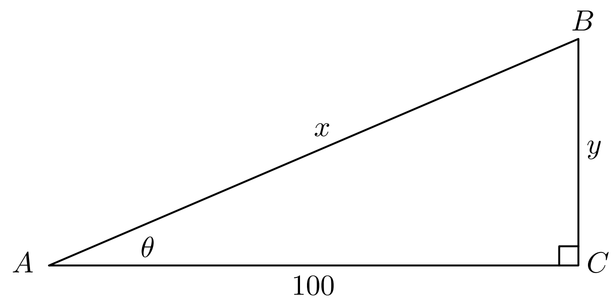
b) Determine whether the lines tangent to the curve at the x-intercepts of the curve are parallel. Show the analysis that leads to your conclusion.

c) Find the points on the curve where the lines tangent to the curve are vertical.

654 (1994AB). A circle is inscribed in a square. The circumference of the circle is increasing at a constant rate of 6 inches per second. As the circle expands, the square expands to maintain the condition of tangency.

a) Find the rate at which the perimeter of the square is increasing. Indicate units of measure.

b) At the instant when the area of the circle is 25π square inches, find the rate of increase in the area enclosed between the circle and the square. Indicate units of measure.



655 (1988BC). The figure above represents an observer at point A watching balloon B as it rises from point C. The balloon is rising at a constant rate of 3 meters per second and the observer is 100 meters from point C.

a) Find the rate of change in x at the instant when $y = 50$.

b) Find the rate of change in the area of right triangle BCA at the instant when $y = 50$.

c) Find the rate of change in θ at the instant when $y = 50$.

656 (1990AB). Let f be the function given by $f(x) = \dfrac{ax+b}{x^2-c}$ and that has the following properties.

(i) The graph of f is symmetric to the y-axis.
(ii) $\lim_{x\to 2^+} f(x) = \infty$.
(iii) $f'(1) = -2$.

a) Determine the values of a, b, and c.

b) Write an equation for each vertical and horizontal asymptote of the graph of f.

c) Sketch the graph of f.

657 (1993BC). Let f be a function differentiable throughout its domain and that has the following properties.

(i) $f(x+y) = \dfrac{f(x)+f(y)}{1-f(x)f(y)}$ for all real numbers x and y in the domain of f.
(ii) $\lim_{h\to 0} f(h) = 0$.
(iii) $\lim_{h\to 0} \dfrac{f(h)}{h} = 1$.

a) Show that $f(0) = 0$.

b) Use the definition of the derivative to show that $f'(x) = 1 + [f(x)]^2$. Indicate clearly where properties (i), (ii), and (iii) are used.

It has been said that World War One was a chemist's war and that World War Two was a physicist's war. There are those who say that the next World War, if one should occur, will be a mathematician's war. —*John H. Curtiss*

2.19 Multiple-Choice Problems on Derivatives

658. Let $F(x) = \begin{cases} \dfrac{x^2+x}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$ Which of the following statements are true of F?

I. F is defined at $x = 0$.
II. $\lim\limits_{x \to 0} F(x)$ exists.
III. F is continuous at $x = 0$.

A) I only B) II only C) I, II only D) II, III only E) I, II, and III

659. If $P(x) = (3x+2)^3$ then the third derivative of P at $x = 0$ is

A) 0 B) 9 C) 54 D) 162 E) 224

660. If $F(x) = 3x$ then $F'(5) =$

A) 0 B) $\frac{1}{5}$ C) -5 D) 3 E) $-\frac{1}{5}$

661. The slope of the curve $y^3 - xy^2 = 4$ at the point where $y = 2$ is

A) -2 B) $\frac{1}{4}$ C) $-\frac{1}{2}$ D) $\frac{1}{2}$ E) 2

662. If $F(x) = x/(x-1)^2$ then the set of all x for which $F(x)$ exists is

A) all real numbers B) $\{x|x \neq -1\}$ C) $\{x|x \neq \frac{1}{3}\}$ D) $\{x|x \neq \pm 1\}$ E) $\{x|x \neq 1\}$

663. If $\lim\limits_{x \to b} G(x) = K$, then which of the following must be true?

A) $G'(b)$ exists.
B) $G(x)$ is continuous at $x = b$.
C) $G(x)$ is defined at $x = b$.
D) $G(b) = K$.
E) None of the above must be true.

664. Which of the following functions are continuous for all real numbers x?

I. $y = x^{4/3}$ II. $y = \sqrt[3]{3x-1}$ III. $y = \dfrac{3x-1}{4x^2+5}$

A) None of these B) I only C) II only D) I, II only E) I, II, and III

665. The equation of the tangent line to the curve $y = x^2 - 4x$ at the point where the curve crosses the y-axis is

A) $y = 8x - 4$ B) $y = -4x$ C) $y = -4$ D) $y = 4x$ E) $y = 4x - 8$

666. The tangent to the curve $y = 2xe^{-x}$ is horizontal when $x =$

A) -2 B) 1 C) -1 D) $\frac{1}{e}$ E) None of the above

If you have an unpleasant nature and dislike people, that is no obstacle to work. —*J. G. Bennett*

667. If $y = \ln\left(\dfrac{e^x}{e^x - 10}\right)$, then $\dfrac{dy}{dx} =$

A) $x - \dfrac{e^x}{e^x - 10}$ B) $-\dfrac{1}{e^x}$ C) $\dfrac{10}{10 - e^x}$ D) 0 E) $\dfrac{e^x - 20}{e^x - 10}$

668. If $y = \ln(x\sqrt{x^2+1})$, then $\dfrac{dy}{dx} =$

A) $1 + \dfrac{x}{x^2+1}$ B) $1 + \dfrac{1}{x\sqrt{x^2+1}}$ C) $\dfrac{2x^2+1}{x\sqrt{x^2+1}}$ D) $\dfrac{2x^2+1}{x(x^2+1)}$ E) $\dfrac{x^2+x+1}{x(x^2+1)}$

669. If $y = e^{-x}\ln x$ then $\dfrac{dy}{dx}$ when $x = 1$ is

A) 0 B) Does not exist C) $\dfrac{2}{e}$ D) $\dfrac{1}{e}$ E) e

670. The slope of the line tangent to the graph of $y = \ln x^2$ at $x = e^2$ is

A) $\dfrac{1}{e^2}$ B) $\dfrac{2}{e^2}$ C) $\dfrac{4}{e^2}$ D) $\dfrac{1}{e^4}$ E) $\dfrac{4}{e^4}$

671. If $y = \ln(x^2 + y^2)$ then the value of $\dfrac{dy}{dx}$ at $(1, 0)$ is

A) 0 B) -1 C) 1 D) 2 E) undefined

672. If $z = \dfrac{3w}{\cos w}$, then $\dfrac{dz}{dw} =$

A) $-\dfrac{3}{\sin w}$ B) $\dfrac{3\cos w - 3w\sin w}{\cos^2 w}$ C) $\dfrac{3}{\sin w}$ D) $\dfrac{3\cos w + 3w\sin w}{\cos^2 w}$ E) None of the above

673. Find the derivative of $y = \dfrac{1}{2\sin 2x}$.

A) $-\csc 2x\cot 2x$ B) $-\csc^2 2x$ C) $-4\csc 2x\cot 2x$ D) $\dfrac{\cos 2x}{2\sqrt{\sin 2x}}$ E) $4\sec 2x$

674. If $y = \sec^2\sqrt{x}$ then $\dfrac{dy}{dx} =$

A) $\dfrac{\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}$ B) $\dfrac{\tan\sqrt{x}}{\sqrt{x}}$ C) $2\sec\sqrt{x}\tan^2\sqrt{x}$ D) $\dfrac{\sec^2\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}$ E) $2\sec^2\sqrt{x}\tan\sqrt{x}$

675. If $y = \sin 11x\cos 11x$, then the derivative of y is

A) $11\cos 11x$ B) $11\cos 22x$ C) $\sin^2 11x - \cos^2 11x$ D) $-121\sin^2 11x$ E) $-121\sin 11x\cos 11x$

A.P. Calculus Test Two
Section One
Multiple-Choice
Calculators Allowed

Time—45 minutes

Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where C is the number of correct responses and I is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1. $\displaystyle\lim_{x\to\infty} \frac{5x^2}{3x^2 + 100000x} =$

A) 0

B) 0.005

C) 1

D) 1.667

E) does not exist

---

2. Which of the following functions are not differentiable at $x = \frac{2}{3}$?

I. $f(x) = \sqrt[3]{x-2}$ II. $g(x) = |3x - 2|$ III. $h(x) = |9x^2 - 4|$

A) I only

B) II only

C) I and II only

D) II and III only

E) I and III only

---

3. If $y = (\ln x)^3$, then $dy/dx =$

A) $\frac{3}{x}(\ln x)^2$

B) $3(\ln x)^2$

C) $3x(\ln x)^2 + (\ln x)^3$

D) $3(\ln x + 1)$

E) None of these

4. If $F(x) = x \sin x$, then find $F'(3\pi/2)$.

A) 0

B) 1

C) -1

D) $3\pi/2$

E) $-3\pi/2$

---

5. The approximate equation of the tangent line to $f(x) = \cos^2(3x)$ at $x = \pi/18$ is

A) $y = -2.598x + 1.203$

B) $y = 2.598x - 1.203$

C) $y = -2.598x + 0.575$

D) $y = 2.598x - 0.575$

E) None of these

---

6. The slope of the tangent to the curve $y^3x + y^2x^2 = 6$ at the point $(2, 1)$ is

A) $-\frac{3}{2}$

B) -1

C) $-\frac{5}{14}$

D) $-\frac{3}{14}$

E) 0

7. Which of the following functions has a derivative at $x = 0$?

$$\begin{array}{rl} \text{I.} & y = \arcsin(x^2 - 1) - x \\ \text{II.} & y = x|x| \\ \text{III.} & y = \sqrt{x^4} \end{array}$$

A) I only

B) II only

C) III only

D) II and III only

E) I, II, and III

---

8. When a wholsale produce market has x crates of lettuce available on a given day, it charges p dollars per crate as determined by the supply equation $px - 20p - 6x + 40 = 0$. If the daily supply is decreasing at the rate of 8 crates per day, at what rate is the price changing when the supply is 100 crates?

A) not changing

B) increasing at \$0.10 per day

C) decreasing at \$0.10 per day

D) increasing at \$1.00 per day

E) decreasing at \$1.00 per day

---

9. Suppose a particle is moving along a coordinate line and its position at time t is given by $s(t) = \dfrac{9t^2}{t^2 + 2}$. For what value of t in the interval $[1, 4]$ is the instantaneous velocity equal to the average velocity?

A) 2.00

B) 2.11

C) 2.22

D) 2.33

E) 2.44

10. A tangent line drawn to the graph of $y = \dfrac{4x}{1+x^3}$ at the point $(1,2)$ forms a right triangle with the coordinate axes. The area of the triangle is

A) 3

B) 3.5

C) 4

D) 4.5

E) 5

---

11. The function

$$f(x) = \begin{cases} 4 - x^2 & x \le 1 \\ mx + b & x > 1 \end{cases}$$

is continuous and differentiable for all real numbers. What must be the values of m and b ?

A) $m = 2,\ b = 1$

B) $m = 2,\ b = 5$

C) $m = -2,\ b = 1$

D) $m = -2,\ b = 5$

E) None of these

---

12. If $f(x) = -x^2 + x$, then which of the following expressions represents $f'(x)$?

A) $\displaystyle\lim_{h\to 0} \frac{(-x^2 + x + h) - (-x^2 + x)}{h}$

B) $\displaystyle\lim_{h\to x} \frac{(-x^2 + x + h) - (-x^2 + x)}{h}$

C) $\displaystyle\frac{\left[-(x+h)^2 + (x+h)\right] - (-x^2 + x)}{h}$

D) $\displaystyle\lim_{h\to 0} \frac{\left[-(x+h)^2 + (x+h)\right] - (-x^2 + x)}{h}$

E) None of these

13. All the functions below, except one, have the property that $f(x)$ is equal to its fourth derivative, $f^{(4)}(x)$. Which one does not have this property?

A) $f(x) = \sin x$

B) $f(x) = \cos x$

C) $f(x) = -5e^x$

D) $f(x) = e^{2x}$

E) $f(x) = e^{-x}$

---

14. If $g(t) = \dfrac{\ln t}{e^t}$, then $g'(t) =$

A) $\dfrac{1 - \ln t}{e^t}$

B) $\dfrac{1 - t \ln t}{e^t}$

C) $\dfrac{t \ln t - 1}{te^t}$

D) $\dfrac{1 - t \ln t}{te^t}$

E) $\dfrac{1 - e^t \ln t}{e^{2t}}$

---

15. If $H(x) = x^3 - x^2 + \dfrac{1}{x}$, which of the following is $H''(2)$?

A) $\frac{31}{4}$

B) $\frac{39}{4}$

C) $\frac{79}{8}$

D) $\frac{81}{8}$

E) $\frac{41}{4}$

A.P. Calculus Test Two
Section Two
Free-Response
No Calculators

Time—45 minutes

Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $y = x^2$ may not be written as `Y1=X^2`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

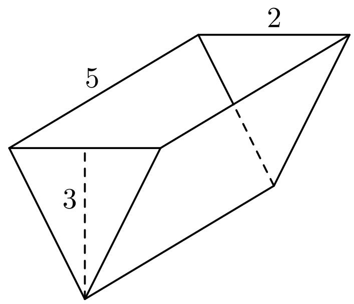
NAME:

1. Consider the curve defined by the equation $y + \cos y = x + 1$ for $0 \le y \le 2\pi$.

a) Find dy/dx in terms of y.

b) Write an equation for each vertical tangent to the curve.

c) Find $\dfrac{d^2y}{dx^2}$ in terms of y.



2. The trough shown in the figure above is 5 feet long and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at the rate of 2 cubic feet per minute. At any time t, let h be the depth and V be the volume of water in the trough.

a) Find the volume of water when the trough is full.

b) What is the rate of change in h at the instant when the trough is $\frac{1}{4}$ full by volume?

c) What is the rate of change in the area of the surface of the water at the instant when the trough is $\frac{1}{4}$ full by volume?

---

3. Let f be the function given by $f(x) = \sqrt{x^4 - 16x^2}$.

a) Find the domain of f.

b) Determine whether f is an odd or even function.

c) Find $f'(x)$.

d) Find the slope of the line normal to the graph of f at $x = 5$.

CHAPTER 3

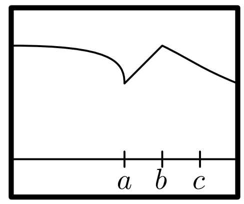
APPLICATIONS of DERIVATIVES

3.1 The Extreme Value Theorem

In the four problems below, match the table with the graph.

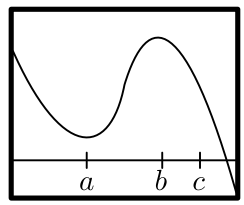
676.

x	$f'(x)$
a	0
b	0
c	5

A) 

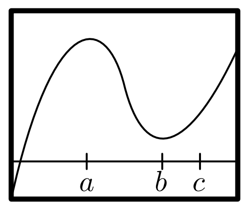
677.

x	$f'(x)$
a	0
b	0
c	-5

B) 

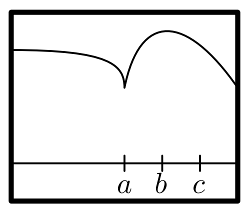
678.

x	$f'(x)$
a	does not exist
b	0
c	-2

C) 

679.

x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7

D) 

680. Let $f(x) = (x-2)^{2/3}$.

a) Does $f'(2)$ exist?

b) Show that the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (b) contradict the Extreme Value Theorem?

d) Repeat parts (a) and (b) for $f(x) = (x-k)^{2/3}$, replacing 2 with k.

681. Let $f(x) = |x^3 - 9x|$.

a) Does $f'(0)$ exist?

b) Does $f'(3)$ exist?

c) Does $f'(-3)$ exist?

d) Determine all extrema of f.

682. The function $V(x) = x(10-2x)(16-2x)$ models the volume of a box. What is the domain of this function? What are the extreme values of V?

3.2 Rolle to the Extreme with the Mean Value Theorem

In the following four problems, verify the three conditions required by Rolle's Theorem and then find a suitable number c guaranteed to exist by Rolle's Theorem.

683. $f(x) = 2x^2 - 11x + 15$ on $[\frac{5}{2}, 3]$

684. $g(x) = x^3 + 5x^2 - x - 5$ on $[-5, -1]$

685. $p(x) = 4x^{4/3} - 6x^{1/3}$ on $[0, 6]$

686. $k(x) = \dfrac{x^2 - 4}{x^2 + 4}$ on $[-2, 2]$

In the following eight problems, verify the two conditions required by the Mean Value Theorem and then find a suitable number c guaranteed to exist by the Mean Value Theorem.

687. $f(x) = 4x^2 - x - 6$ on $[1, 3]$

688. $g(x) = \dfrac{x-1}{x+2}$ on $[0, 2]$

689. $p(x) = 3x^{2/3} - 2x$ on $[0, 1]$

690. $k(x) = x^4 - 3x$ on $[1, 3]$

691. $F(x) = x^3$ on $[1, 3]$

692. $G(x) = (x-1)^3$ on $[-1, 2]$

693. $P(x) = x^2 + 5x$ on $[0, 2]$

694. $H(x) = x^3$ on $[-1, 3]$

Find critical points of the functions in the following four problems.

695. $f(x) = 3x^2 - 5x + 1$

696. $h(x) = x^4 - 2x^2 + 3$

697. $p(x) = \dfrac{3x-2}{x-4}$

698. $h(x) = 2x^{5/3} - x^{2/3} + 3$

699. The function $f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & x = 1 \end{cases}$ is zero at $x = 0$ and $x = 1$, and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. Doesn't this contradict Rolle's Theorem?

700. A trucker handed in a ticket at a toll booth showing that in 2 hours he had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

I advise my students to listen carefully the moment they decide to take no more mathematics courses. They might be able to hear the sound of closing doors. —*James Caballero*

3.3 The First and Second Derivative Tests

For the following, find: a) the domain of each function, b) the x-coordinate of the local extrema, and c) the intervals where the function is increasing and/or decreasing.

701. $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x - 1$

702. $g(x) = x^3 - 5x^2 - 8x$

703. $h(x) = x + \dfrac{4}{x}$

704. $p(x) = \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

705. $h(x) = (2-x)^2(x+3)^3$

706. $m(x) = 3x\sqrt{5-x}$

707. $f(x) = x^{2/3}(x-5)^{-1/3}$

708. $h(x) = \frac{1}{7}x^{7/3} - x^{4/3}$

709. Find the values of a and b so that the function $f(x) = \frac{1}{3}x^3 + ax^2 + bx$ will have a relative extreme at $(3,1)$.

710. Find the values of a, b, c, and d so that the function $f(x) = ax^3 + bx^2 + cx + d$ will have relative extrema at $(-1,1)$ and $(-2,4)$.

In the following problems, find a) the coordinates of inflection points and b) the intervals where the function is concave up and/or concave down.

711. $g(x) = x^3 - 5x$

712. $h(x) = 2x^3 - 3x^2 - 8x + 1$

713. $h(x) = (3x+2)^3$

714. $p(x) = \dfrac{3}{x^2+4}$

715. $f(x) = \begin{cases} x^2 - 3 & x > 3 \\ 15 - x^2 & x \le 3 \end{cases}$

716. $p(x) = \begin{cases} 2x^2 & x \ge 0 \\ -2x^2 & x < 0 \end{cases}$

717. Determine the values of a and b so that the function $p(x) = ax^4 + bx^3$ will have a point of inflection at $(-1,3)$.

718. Determine the values of a, b, and c so that the function $p(x) = ax^3 + bx^2 + cx$ will have an inflection point at $(-1,3)$ and the slope of the tangent at $(-1,3)$ will be -2.

The calculus is the greatest aid we have to the application of physical truth in the broadest sense of the word. —*W. F. Osgood*

3.4 Derivatives and Their Graphs

719. The graph of a function f is given below. Estimate the values of $f'(x)$ at the following points.

a) $x = -2$

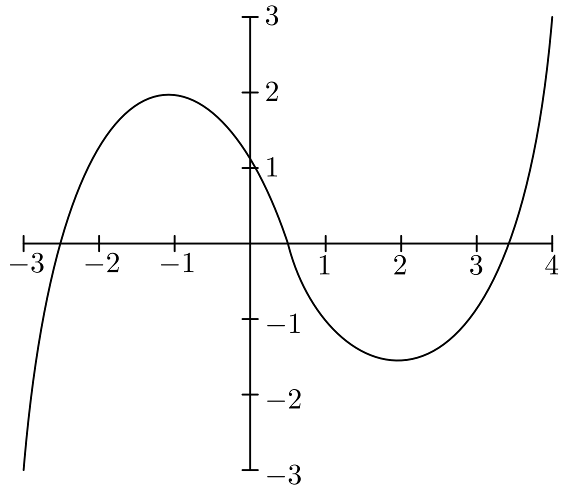
b) $x = -1$

c) $x = 0$

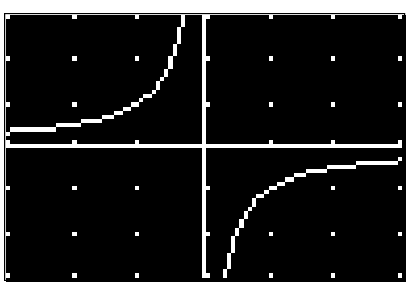
d) $x = 1.5$

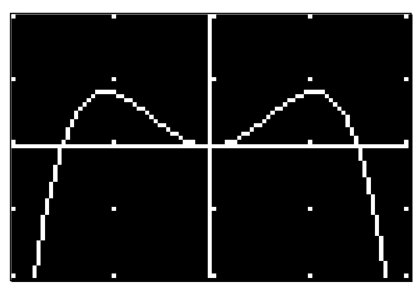
e) $x = 2$

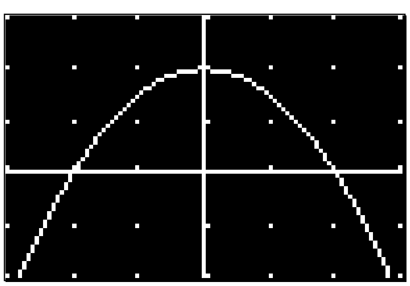
f) $x = 3$

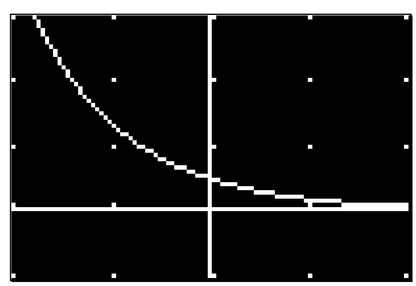


720. Sketch the graphs of the derivatives of the four functions shown below.

a) 

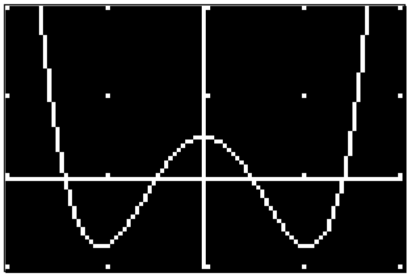
c) 

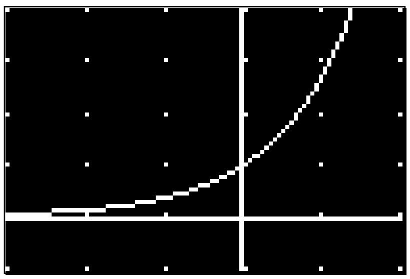
b) 

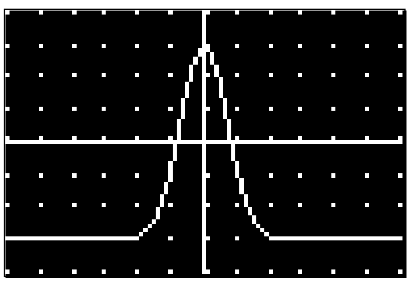
d) 

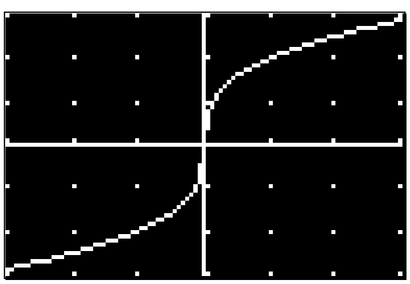
It seems to me that we are all afflicted with an urge and possessed with a longing for the impossible. The reality around us, the three-dimensional world surrounding us, is too common, too dull, too ordinary for us. We hanker after the unnatural or supernatural, that which does not exist, a miracle. —*M. C. Escher*

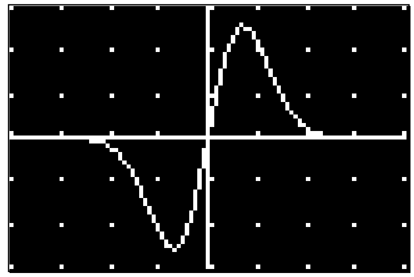
721. The graphs of some functions are given below. Indicate on what intervals the functions are increasing and on what intervals the functions are decreasing, and then sketch the graphs of their derivatives.

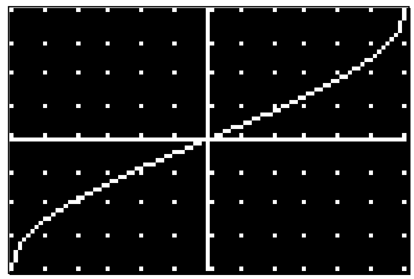
a) 

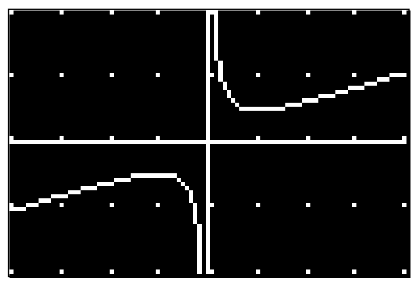
b) 

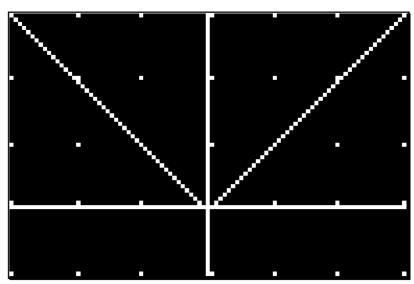
c) 

d) 

e) 

f) 

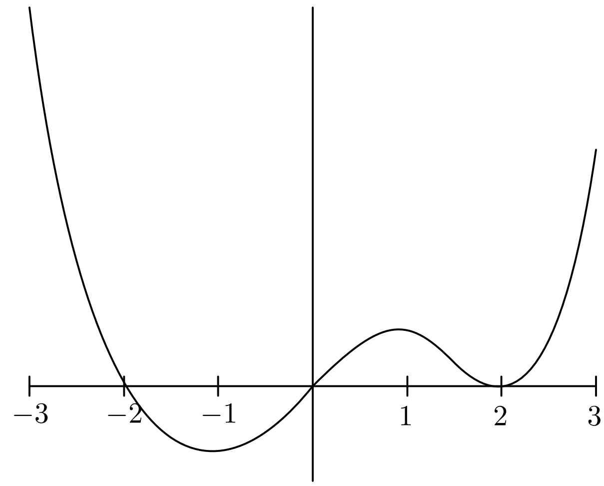
g) 

h) 

The difference between ordinary and extraordinary is that little extra. —*Anonymous*

3.5 Two Derivative Problems

722 (AP). The graph below is the graph of the derivative of a function f.

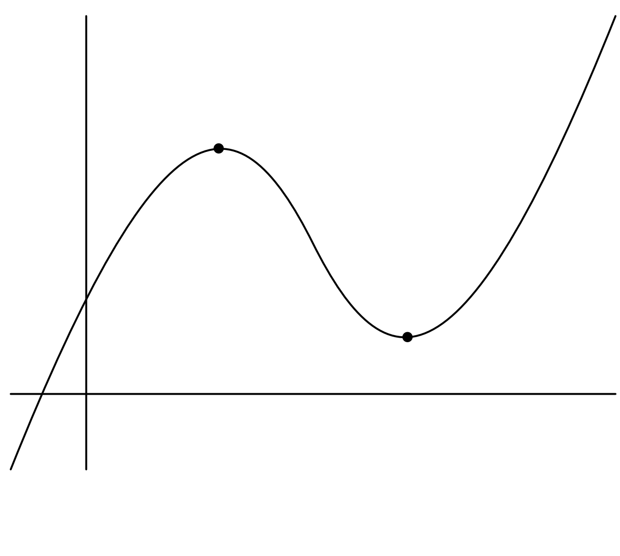


a) Find where f is increasing and where it is decreasing.

b) Find all local maxima and local minima of f.

c) If $f(-3) = -2$, sketch the graph of f.

723 (AP). The graph below is that of a function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are constants. Show that the x-coordinates of the two marked points are given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 3ac}}{3a}$.



In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting President used the third derivative to advance his case for re-election. —*Hugo Rossi*

3.6 Sketching Functions

For the following six problems, find:
a) the domain
b) the zeros
c) the y-intercept
d) coordinates of local extrema
e) intervals where the function increases and/or decreases
f) coordinates of inflection points
g) intervals where the function is concave up and/or concave down, and then
h) sketch the graph of the function.

724. $h(x) = (x-1)^3(x-5)$

725. $f(x) = (x-2)^{1/3} - 4$

726. $n(x) = \dfrac{3x^2}{x^2-9}$

727. $f(x) = x^2e^x$

728. $j(x) = x\ln x$

729. $p(x) = \dfrac{\ln x}{x}$

730. Sketch a graph of a function whose derivative satisfies the properties given in the following table.

x	$(-\infty,-1)$	-1	$(-1,1)$	1	$(1,3)$	3	$(3,\infty)$
$f'(x)$	positive	0	negative	0	negative	0	positive

731. Suppose f has a continuous derivative whose values are given in the following table.

x	0	1	2	3	4	5	6	7	8	9	10
$f'(x)$	5	2	1	-2	-5	-3	-1	2	3	1	-1

a) Estimate the x-coordinates of critical points of f for $0 \le x \le 10$.

b) For each critical point, indicate if it is a local maximum of f, local minimum, or neither.

732. Suppose f is a continuous and differentiable function on the interval $[0,1]$ and $g(x) = f(3x)$. The table below gives some values of f.

x	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1.01	1.042	1.180	1.298	1.486	1.573

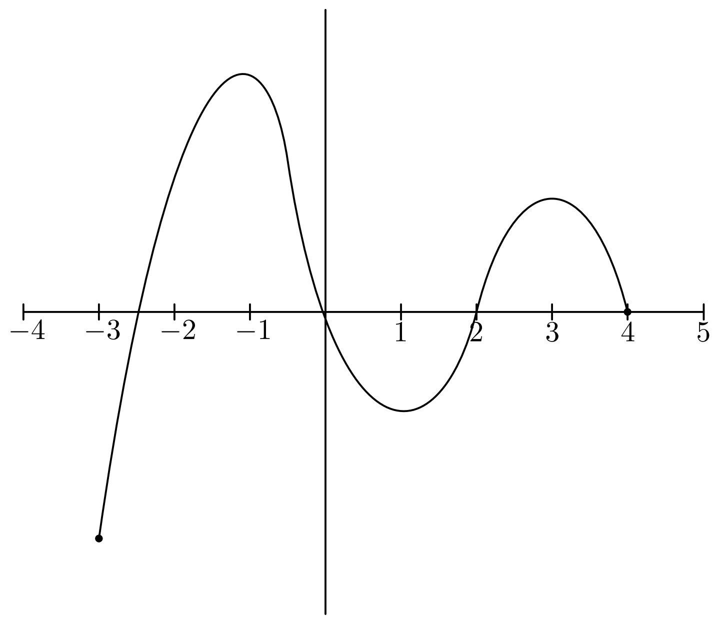
What is the approximate value of $g'(0.1)$?

733. The figure below shows the graph of $g'(x)$, the derivative of a function g, with domain $[-3, 4]$.

a) Determine the values of x for which g has a relative minimum and a relative maximum. Justify your answer.

b) Determine the values of x for which g is concave down and concave up. Justify your answer.

c) Based on the information given and the fact that $g(-3) = 3$ and $g(4) = 6$, sketch a possible graph of g.



Bistromathics itself is simply a revolutionary new way of understanding the behavior of numbers. Just as Einstein observed that space was not an absolute but depended on the observer's movement in space, and that time was not an absolute, but depended on the observer's movement in time, so it is now realized that numbers are not absolute, but depend on the observer's movement in restaurants.

The first nonabsolute number is the number of people for whom the table is reserved. This will vary during the course of the first three telephone calls to the restaurant, and then bear no apparent relation to the number of people who actually turn up, or to the number of people who subsequently join them after the show/match/party/gig, or to the number of people who leave when they see who else has turned up.

The second nonabsolute number is the given time of arrival, which is now known to be one of the most bizarre of mathematical concepts, a "recipriversexcluson", a number whose existence can only be defined as being anything other than itself. In other words, the given time of arrival is the one moment of time at which it is impossible that any member of the party will arrive. Recipriversexclusons now play a vital part in many branches of math, including statistics and accountancy and also form the basic equations used to engineer the Somebody Else's Problem field.

The third and most mysterious piece of nonabsoluteness of all lies in the relationship between the number of items on the bill, the cost of each item, the number of people at the table and what they are each prepared to pay for. (The number of people who have actually brought any money is only a subphenomenon of this field.)
—*Douglas Adams,* Life, the Universe, and Everything

3.7 Problems of Motion

734. A car is moving along Highway 20 according to the given equation, where x meters is the directed distance of the car from a given point P at t hours. Find the values of t for which the car is moving to the right and when it is moving to the left. Draw a diagram to describe the motion of the car.

a) $x = 2t^3 + 15t^2 + 36t + 2$

b) $x = 2t^3 + 9t^2 - 60t - 7$

735. A car is moving along Highway 138 according to the given equation, where x meters is the directed distance of the car from a given point P at t hours. Find the values of t for which the acceleration is zero, and then find the position of the car at this time.

a) $x = \frac{1}{4}t^4 + \frac{1}{6}t^3 - t^2 + 1$

b) $x = -3\sqrt{t} - \dfrac{1}{12\sqrt{t}}$ for $t > 0$

736. A snail moves along the x-axis so that at time t its position is given by $x(t) = 3\ln(2t-5)$, for $t > \frac{5}{2}$.

a) What is the position and the velocity of the snail at time $t = 3$?

b) When is the snail moving to the right, and when is it moving to the left?

737. An ant moves along the x-axis so that at time t its position is given by $x(t) = 2\cos\left(\frac{\pi}{2}t^2\right)$, for values of t in the interval $[-1, 1]$.

a) Find an expression for the velocity of the ant at any given time t.

b) Find an expression for the acceleration at any given time t.

c) Determine the values of t for which the ant is moving to the right. Justify your answer.

d) Determine the values of t for which the ant changes direction. Justify your answer.

738. A particle is moving along the x-axis so that its position is given by

$$x(t) = \frac{3\pi}{2}t^2 - \sin\left(\frac{3\pi}{2}t^2\right),$$

for $0 < t \leq 2$.

a) Find an expression for the velocity of the particle at any given time t.

b) Find an expression for the acceleration at any given time t.

c) Find the values of t for which the particle is at rest.

d) Find the position of the particle at the time(s) found in part c).

Thus metaphysics and mathematics are, among all the sciences that belong to reason, those in which imagination has the greatest role. I beg pardon of those delicate spirits who are detractors of mathematics for saying this .... The imagination in a mathematician who creates makes no less difference than in a poet who invents.... Of all the great men of antiquity, Archimedes may be the one who most deserves to be placed beside Homer. —*Jean le Rond d'Alembert*

739. At time $t \geq 0$, the velocity of a body moving along the x-axis is $v(t) = t^2 - 4t + 3$.

a) Find the body's acceleration each time the velocity is zero.

b) When is the body moving forward? Backward?

c) When is the body's velocity increasing? Decreasing?

740. The position of a ball moving along a straight line is given by $s(t) = \frac{4}{3}e^{3t} - 8t$.

a) Write an expression for the velocity at any given time t.

b) Write an expression for the acceleration at any given time t.

c) Find the values of t for which the ball is at rest.

d) Find the position of the ball at the time(s) found in part c).

741. A racehorse is running a 10 furlong race (1 furlong is 220 yards). As the horse passes each furlong marker, F, a steward records the time elapsed, t, since the beginning of the race, as shown in the table below.

F	0	1	2	3	4	5	6	7	8	9	10
t	0	20	33	46	59	73	86	100	112	124	135

a) How long does it take the horse to finish the race?

b) What is the average speed of the horse over the the first 5 furlongs?

c) What is the approximate speed of the horse as it passes the 3-furlong marker?

d) During which portion of the race is the horse running the fastest? Accelerating the fastest?

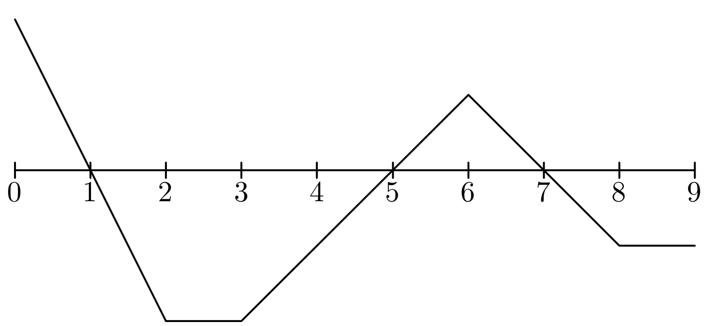
742. The graph below shows the velocity $v = f(t)$ of a particle moving on a coordinate line.

a) When does the particle move forward? move backward? speed up? slow down?

b) When is the particle's acceleration positive? negative? zero?

c) When does the particle move at its greatest speed?

d) When does the particle stand still for more than an instant?



Every job is a little self-portrait of the person who did it. —*Anonymous*

3.8 Maximize or Minimize?

743. The famous Kate Lynn Horsefeed is building a box as part of her science project. It is to be built from a rectangular piece of cardboard measuring 25 cm by 40 cm bycutting out a square from each corner and then bending up the sides. Find the size of the corner square which will produce a container that will hold the most.

744. Ashley is building a window in the shape of an equilateral triangle whose sides each measure 4 meters. Ashley wants to inscribe a rectangular piece of stained glass in the triangle, so that two of the vertices of the rectangle lie on one of the sides of the triangle. Find the dimensions of the rectangle of maximum area that can be inscribed in the given triangle.

745. It has been determined by the brilliant deductive mind of Bruce Wayne that Gotham Highway is located on the line $y = 2x + 3$. Determine the point on Gotham Highway closest to the Wayne Foundation Building, which happens to be located at the point $(1, 2)$.

746. Vaidehi wants to cut a 30-meter piece of iron into two pieces. One of the pieces will be used to build an equilateral triangle, and the other to build a rectangle whose length is three times its width. Where should Vaidehi cut the iron bar if the combined area of the triangle and the rectangle is to be a minimum? How could the combined area of these two figures be a maximum? Justify your answers.

747. An open oak wood box with a square base is to be constructed using 192 cm^2 of oak. If the volume of the box is to be maximized, find its dimensions.

748. At the Skywalker moisture farm on the desert planet Tatooine, there are 24 moisture processors, with an average yield per processor of 300 cubits of moisture. Research conducted at Mos Eisley University concludes that when an additional processor is used, the average yield per processor is reduced by 5 cubits. Help Owen and Beru Skywalker find the number of moisture processors that will maximize the number of cubits.

749. The fence around Wayne Manor is going to be replaced. No fence will be required on the side lying along Gotham River. If the new wrought iron fence costs \$12 per meter for the side parallel to the river, and \$4 per meter for the other two sides, find the dimensions of the maximum area that can be enclosed by the fence if Bruce Wayne cannot spend more than \$3600.

750. The Gotham-Metropolis Highway is a toll road that has averaged 54,000 cars per day over the past five years, with a \$.50 charge per car. A study conducted by the Ray Chulldel Lavet University concludes that for every \$.05 increase in the toll, the number of cars will be reduced by 500. In order to maximize revenue, what toll should the highway charge?

751. The range R of a projectile whose muzzle velocity in meters per second is v, and whose angle of elevation in radians is θ, is given by $R = (v^2 \sin(2\theta))/g$ where g is the acceleration of gravity. Which angle of elevation gives the maximum range of the projectile?

752. A piece of wire 100 cm long is to be cut into several pieces and used to construct the skeleton of a box with a square base.

a) What is the largest possible volume that such a box can have?

b) What is the largest possible surface area?

753. In medicine, the reaction $R(x)$ to a dose x of a drug is given by $R(x) = Ax^2(B-x)$, where $A > 0$ and $B > 0$. The sensitivity $S(x)$ of the body to a dose of size x is defined to be $R'(x)$. Assume that a negative reaction is a bad thing.

a) What seems to be the domain of R? What seems to be the physical meaning of the constant B? What seems to be the physical meaning of the constant A?

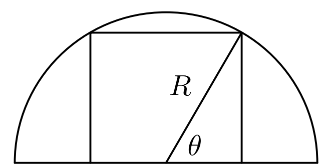
b) For what value of x is R a maximum?

c) What is the maximum value of R?

d) For what value of x is the sensitivity a minimum?

e) Why is it called sensitivity?

754. What is the area of the largest rectangle that can be inscribed in a semicircle of radius R so that one of the sides of the rectangle lies on the diameter of the semicircle?



755. An electronics store needs to order a total of 2400 CD players over the course of a year. It will receive them in several shipments, each containing an equal number of CD players. The shipping costs are \$50 for each shipment, plus a yearly fee of \$2 for each CD player in a single shipment. What size should each shipment be in order to minimize yearly shipping costs?

756. A rectangle in the first quadrant has one side on the y-axis, another on the x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What is the maximum area of the rectangle?

757. The positions of two particles on the x-axis are $x_1 = \sin t$ and $x_2 = \sin(t + \frac{\pi}{3})$.

a) At what time(s) in the interval $[0, 2\pi]$ do the particles meet?

b) What is the farthest apart that the particles ever get?

c) When in the interval $[0, 2\pi]$ is the distance between the particles changing the fastest?

758. One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2}$$

where q is the quantity ordered when things run low, k is the cost of placing an order (a constant), m is the number of items sold each week (a constant), h is the weekly holding cost per item (a constant), and c is a constant. What is the quantity that will minimize $A(q)$? (The expression you get for your answer is called the *Wilson lot size formula.*)

759. The function $f(x) = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $[0, \pi]$. Find its exact value.

When introduced at the wrong time or place, good logic may be the worst enemy of good teaching. —*George Polya*

3.9 More Tangents and Derivatives

FIND THE TANGENT LINES TO EACH OF THE FOLLOWING AT $x = 0$.

760. $\sin x$

761. $\cos x$

762. $\tan x$

763. e^x

764. $\ln(1 + x)$

765. $(1 + x)^k$, for nonzero constant k.

766. $(1 - x)^k$, for nonzero constant k.

767. Using the tangent lines found above, approximate the values of $\sin 0.1$; $\cos 0.1$; $\tan 0.1$; $e^{0.1}$; $\ln(1.1)$; $(1.1)^5$; and $(0.9)^4$.

768. As noted in problems 765 and 766, k is any nonzero constant. Using the tangent found above, approximate $\sqrt{1.06}$; $\sqrt[3]{1.06}$; $\frac{1}{1.06}$; and $\frac{1}{(1.06)^2}$. Then, using your calculator, determine the difference in the approximation compared to the more accurate value given by the calculator.

769. Let $f'(x) = (x - 1)e^{-x}$ be the derivative of a function f. What are the critical points of f? On what intervals is f increasing or decreasing? At what points, if any, does f have local extrema?

770. Let $f'(x) = (x - 1)^2(x - 2)$ be the derivative of a function f. What are the critical points of f? On what intervals is f increasing or decreasing? At what points, if any, does f have local extrema?

771. Let f be a continuous function on $[0, 3]$ that has the following signs and values as in the table below.

x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3
$f(x)$	0	positive	2	positive	0	negative	-2
$f'(x)$	3	positive	0	negative	does not exist	negative	-3
$f''(x)$	0	negative	-1	negative	does not exist	negative	0

Find the absolute extrema of f and where they occur; find any points of inflection; and sketch a possible graph of f.

772. A particle moves along the x-axis as described by $x(t) = 3t^2 - 2t^3$. Find the acceleration of the particle at the time when the velocity is a maximum.

773. Find the values of a, b, c, and d such that the cubic $f(x) = ax^3 + bx^2 + cx + d$ has a relative maximum at $(2, 4)$, a relative minimum at $(4, 2)$, and an inflection point at $(3, 3)$.

774. Show that the point of inflection of $f(x) = x(x - 6)^2$ lies midway between the relative extrema of f.

3.10 More Excitement with Derivatives!

775. Let $f(x) = |x| + x$. Does $f'(0)$ exist? Explain.

776. Determine whether the following functions have a derivative at $x = 0$.

a) $f(x) = x|x|$ b) $f(x) = x^2|x|$ c) $f(x) = x^3|x|$ d) $f(x) = x^4|x|$

777. Use the definition of the derivative to find $g'(1)$: a) $g(x) = 2x^2 + 3x$; b) $g(x) = \dfrac{1}{2x+1}$.

778. Find $\dfrac{dy}{dx}$ for each of the following.

a) $y = 2x^{1/3}$

b) $y = 5x^{11}$

c) $y = x \arctan x$

d) $y = \frac{1}{2}x^{-3/4}$

e) $y = 25x^{-1} + 12x^{1/2}$

f) $y = (2x - 5)(3x^4 + 5x + 2)$

g) $y = \dfrac{x^2 + 2x - 1}{x^2 - 1}$

779. What is the slope of the curve $y = \dfrac{t}{t+5}$ at the point $t = 2$? What is the equation of the tangent line at this point?

780. What is the slope of the curve $y = \dfrac{t^2}{t^2+1}$ at the point $t = 1$? What is the equation of the tangent line at this point?

781. Consider a function f which satisfies the following properties.

i) $f(x + y) = f(x)f(y)$
ii) $f(0) \neq 0$
iii) $f'(0) = 1$

a) Show that $f(0) = 1$. *Hint:* Let $x = y = 0$ in (i).

b) Show that $f(x) \neq 0$ for all x. *Hint:* Let $y = -x$ in (ii).

c) Use the definition of the derivative to show that $f'(x) = f(x)$ for all real numbers x.

d) There is only one function that satisfies properties (i), (ii), and (iii). Name it.

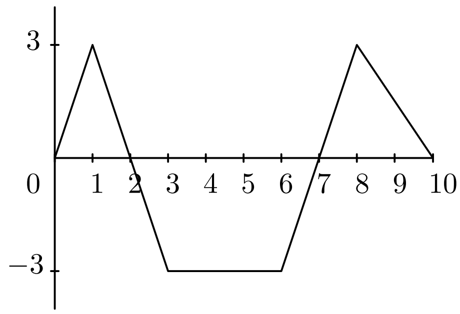
782. If $\sin x = e^y$, then find $\dfrac{dy}{dx}$ in terms of x.

783. Find $\displaystyle\lim_{x\to 0} \frac{e^{3+x} - e^3}{x}$.

We must view with profound respect the infinite capacity of the human mind to resist the introduction of useful knowledge. —*Thomas R. Lounsbury*

3.11 Bodies, Particles, Rockets, Trucks, and Canals

784. The graph below shows the velocity $v(t)$ in meters per second of a body moving along the coordinate line.



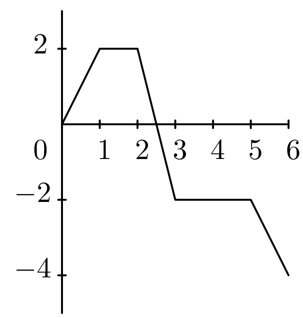
a) When does the body reverse direction?

b) When is the body moving at a constant speed?

c) Graph the body's speed for the interval $[0, 10]$.

d) Graph the acceleration.

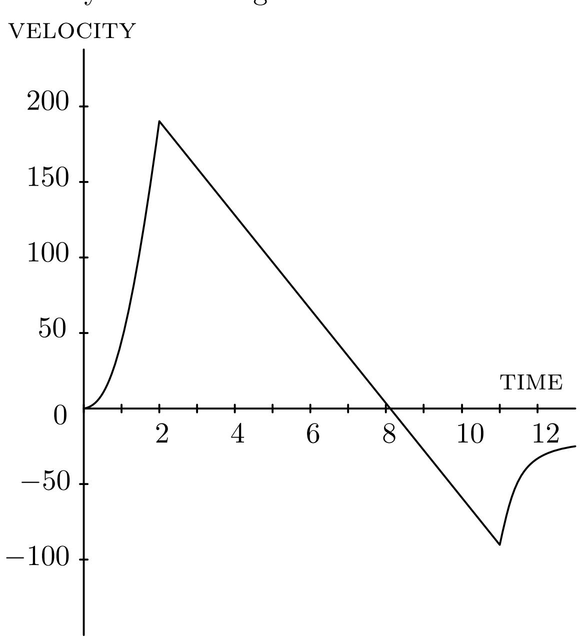
785. A particle P moves along the coordinate line so that the graph at the right is its position $x(t)$ for time t in the interval $[0, 6]$.



a) When is P moving to the left? Moving to the right? Standing still?

b) Graph the particle's velocity and speed.

786. When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands. The figure here shows the velocity from the flight of a model rocket.



a) How fast was the rocket climbing when the engine stopped?

b) For how many seconds did the engine burn?

c) When did the rocket reach its highest point? What was its velocity then?

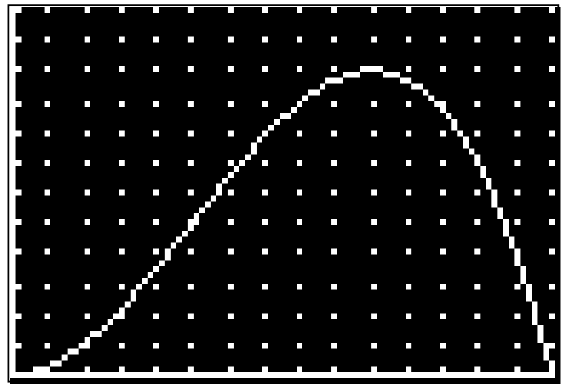
d) When did the parachute pop out? How fast was the rocket falling then?

e) How long did the rocket fall before the parachute opened?

f) When was the rocket's acceleration greatest?

g) When was the acceleration constant? What was its value then?

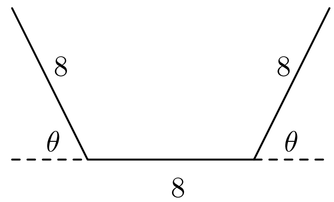
787. The graph shows the position s (for $0 \leq s < 600$) of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 hours later at $t = 15$. (Note: the vertical axis scale is 50, while the horizontal axis scale is 1.)



a) Graph the truck's velocity and acceleration for $0 \leq t \leq 15$.

b) Suppose $s = 15t^2 - t^3$. Graph s' and s'' on your calculator and compare with the graphs obtained in part (a).

788. The cross sections of an irrigation canal are isosceles trapezoids of which three sides are 8 feet long. Determine the angle of elevation θ of the sides so that the area of the cross section is a maximum.



Both of the following problems refer to the graphs below.

789. Let $h(x) = f(x)g(x)$, where the functions f and g are given by the graphs.

a) Estimate $h(-2)$ and $h(3)$.

b) Estimate $f'(-2)$, $f'(3)$, $g'(-2)$, and $g'(3)$.

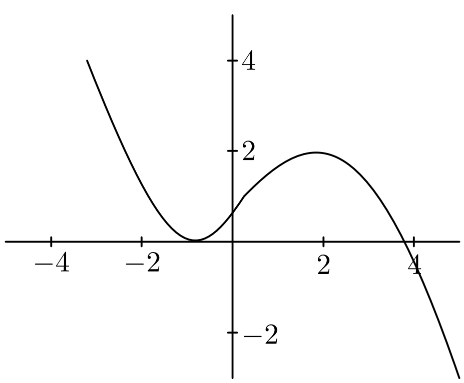
c) Estimate $h'(-2)$ and $h'(3)$.

790. Let $h(x) = f(g(x))$, where the functions f and g are given by the graphs.

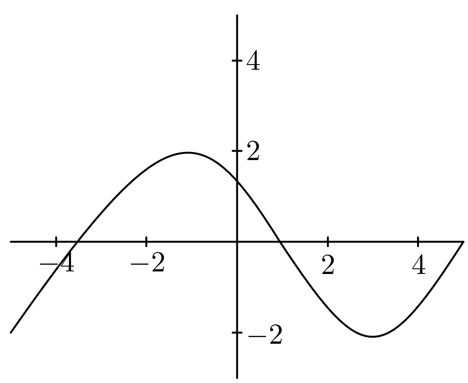
a) Estimate $h(-2)$ and $h(3)$.

b) Is $h'(-3)$ positive, negative, or zero? Explain how you know this.

c) Is $h'(-1)$ positive, negative, or zero? Explain how you know this.



Graph of f



Graph of g

3.12 Even More Excitement with Derivatives!

791. Suppose f and g are differentiable functions for which:

$$\begin{array}{rl} \text{i)} & f(0) = 0 \text{ and } g(0) = 1; \\ \text{ii)} & f'(x) = g(x) \text{ and } g'(x) = -f(x). \end{array}$$

a) Let $h(x) = [f(x)]^2 + [g(x)]^2$. Find $h'(x)$, and use this to show that $[f(x)]^2 + [g(x)]^2 = 1$ for all x.

b) Suppose $F(x)$ and $G(x)$ are another pair of differentiable functions which satisfy properties (i) and (ii) and let $k(x) = [F(x) - f(x)]^2 + [G(x) - g(x)]^2$. Find $k'(x)$ and use this to discover the relationship between $f(x)$ and $F(x)$, and $g(x)$ and $G(x)$.

c) Think of a pair of functions f and g which satisfy properties (i) and (ii). Can there be any others? Justify your answer.

792 (AP). If $x = \left(\dfrac{y^2-1}{3}\right)^3 - \dfrac{y^2-1}{3}$, find $\dfrac{dy}{dx}$ at the point when $y = 2$.

793 (AP). Let $f(x) = x^3 + x$. If h is the inverse function of f, find $h'(2)$.

794 (AP). For $-\frac{\pi}{2} < x < \frac{\pi}{2}$, define $f(x) = \dfrac{x + \sin x}{\cos x}$.

a) Is f an even function, an odd function, or neither? Justify your answer.

b) Find $f'(x)$.

c) Find an equation of the line tangent to the graph of f at the point where $x = 0$.

795 (AP). Find all of the following functions that satisfy the equation $f''(x) = f'(x)$.

a) $f(x) = 2e^x$ b) $f(x) = e^{-x}$ c) $f(x) = \sin x$ d) $f(x) = \ln x$ e) $f(x) = e^{2x}$

796 (AP). If $f(x) = e^x$, which of the following is equal to $f'(e)$?

A) $\displaystyle\lim_{\Delta x \to 0} \frac{e^{x+\Delta x}}{\Delta x}$

B) $\displaystyle\lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^e}{\Delta x}$

C) $\displaystyle\lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e}{\Delta x}$

D) $\displaystyle\lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - 1}{\Delta x}$

E) $\displaystyle\lim_{\Delta x \to 0} e^e \frac{e^{\Delta x} - 1}{\Delta x}$

797 (AP). Let $f(x) = \sin x + \cos x$. Find $(f^{-1})'(\sqrt{2})$.

798 (AP). Let f be a continuous function on $[-3,3]$ whose first and second derivatives have the following signs and values.

x	$-3 < x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$1 < x < 3$
$f'(x)$	positive	0	negative	negative	negative	0	negative
$f''(x)$	negative	negative	negative	0	positive	0	negative

a) What are the x-coordinates of the relative extrema of f on $[-3,3]$?

b) What are the x-coordinates of the points of inflection of f on $[-3,3]$?

c) Sketch a possible graph of f which satisfies all the given properties.

799 (AP). Let f be a function which is twice differentiable for all real numbers and which satisfies the following properties:

i) $f(0) = 1$
ii) $f'(x) > 0$ for all $x \neq 0$
iii) f is concave down for all $x < 0$ and is concave up for all $x > 0$.

Let $g(x) = f(x^2)$.

a) Sketch a possible graph for f which takes into account its properties given above.

b) Find the x-coordinates of all relative minimum points of g. Justify your answer.

c) Where is the graph of g concave up? Justify your answer.

d) Use the information obatined in the three previous parts to sketch a possible graph of g.

For the following six problems, find the domain and coordinates of local extrema.

800. $P(x) = 10^{x^2-1}$

801. $A(x) = 10^{1-x^2}$

802. $T(x) = 10^{1/(x^2-1)}$

803. $H(x) = e^{3x/(x+1)}$

804. $Y(x) = \log(\frac{1}{x})$

805. $A(x) = \log\sqrt{1-x^2}$

For the following three problems, find y'' in simplest factored form.

806. $y = xe^{-x}$

807. $y = x^2e^x$

808. $y = e^{e^x}$

3.13 Sample A.P. Problems on Applications of Derivatives

809. Sketch the graph of a continuous function f with $f(0) = -1$ and $f'(x) = \begin{cases} 1 & x < -1 \\ -2 & x > -1. \end{cases}$

810 (1987BC). Consider the curve given by the equation $y^3 + 3x^2y + 13 = 0$.

a) Find dy/dx.

b) Write an equation for the line tangent to the curve at the point $(2, -1)$.

c) Find the minimum y-coordinate of any point on the curve. Justify your answer.

811 (1990AB). Let f be a function defined by $f(x) = \sin^2 x - \sin x$ for $0 \le x \le \frac{3\pi}{2}$.

a) Find the x-intercept of the graph of f.

b) Find the intervals on which f is increasing.

c) Find the absolute maximum value and the absolute minimum value of f. Justify your answer.

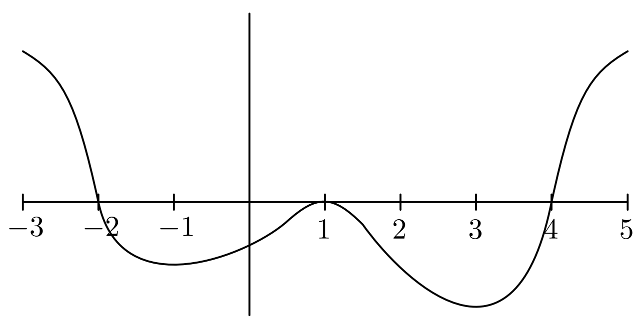
812. Consider the curve $y = x^3 + x$.

a) Find the tangents to the curve at the points where the slope is 4.

b) What is the smallest slope of the curve?

c) At what values x does the curve have the slope found in part (b)?

813 (1996AB). The figure below shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.



a) For what values of x does f have a relative maximum? Why?

b) For what values of x does f have a relative minimum? Why?

c) On what intervals is the graph of f concave upward? Use f' to justify your answer.

d) Suppose that $f(1) = 0$. Draw a sketch of f that shows the general shape of the graph on the open interval $0 < x < 2$.

814 (1992AB). Let f be the function given by $f(x) = \ln\left|\dfrac{x}{1+x^2}\right|$.

a) Find the domain of f.

b) Determine whether f is even, odd or neither. Justify your conclusion.

c) At what values of x does f have a relative maximum or a relative minimum? For each such x, use the first derivative test to determine whether $f(x)$ is a relative maximum or a relative minimum.

d) Find the range of f.

815 (Calculator). Let $f(x) = x\ln x$, $a = 0.5$, and $b = 3$.

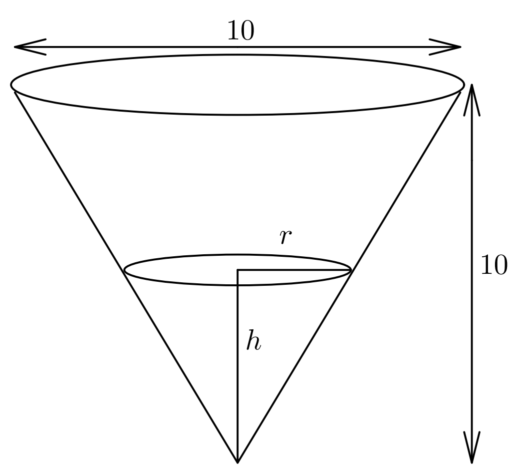
a) Show that f satisfies the hypotheses of the Mean Value Theorem on the interval $[a, b]$.

b) Find the values(s) of c in (a, b) for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

c) Write an equation for the secant line AB where $A = (a, f(a))$ and $B = (b, f(b))$.

d) Write an equation for the tangent line that is parallel to the secant line AB.

816 (2002AB). A container has the shape of an open right circular cone, as shown in the figure below. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.



a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

817 (1992BC). Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & x \le 1 \\ x^2 + kx + p & x > 1. \end{cases}$

a) For what values of k and p will f be continuous and differentiable at $x = 1$?

b) For the value of k and p found in part (a), on what interval or intervals is f increasing?

c) Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

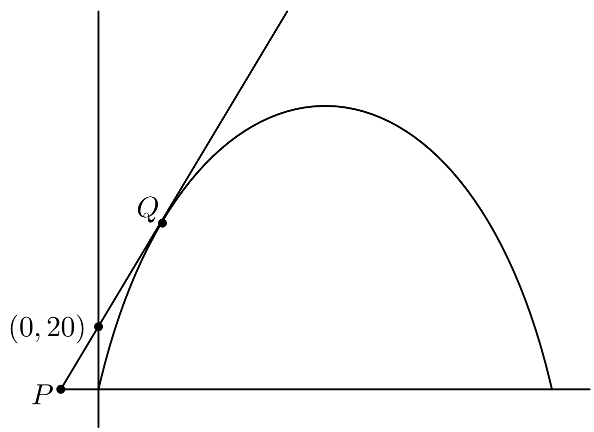
818 (1989BC). Consider the function f defined by $f(x) = e^x \cos x$ with domain $[0, 2\pi]$.

a) Find the absolute maximum and minimum values of $f(x)$.

b) Find intervals on which f is increasing.

c) Find the x-coordinate of each point of inflection of the graph of f.

819 (1996AB). Line ℓ is tangent to the graph of $y = x - \dfrac{x^2}{500}$ at the point Q, as shown in the figure below.



a) Find the x-coordinate of Q.

b) Write an equation for line ℓ.

c) Suppose the graph of y, where x and y are measured in feet, represents a hill. There is a 50-foot tree growing vertically at the top of the hill. Does a spotlight at point P directed along line ℓ shine on any part of the tree? Show the work that leads to your conclusion.

Perhaps the most surprising thing about mathematics is that it is so surprising. The rules which we make up at the beginning seem ordinary and inevitable, but it is impossible to foresee their consequences. These have only been found out by long study, extending over many centuries. Much of our knowledge is due to a comparatively few great mathematicians such as Newton, Euler, Gauss, or Riemann; few careers can have been more satisfying than theirs. They have contributed something to human thought even more lasting than great literature, since it is independent of language. —*E. C. Titchmarsh*

3.14 Multiple-Choice Problems on Applications of Derivatives

820. The value of c guaranteed to exist by the Mean Value Theorem for $V(x) = x^2$ in the interval $[0, 3]$ is

A) 1 B) 2 C) $\frac{3}{2}$ D) $\frac{1}{2}$ E) None of these

821. If $P(x)$ is continuous in $[k, m]$ and differentiable in (k, m), then the Mean Value Theorem states that there is a point a between k and m such that

A) $\dfrac{P(k) - P(m)}{m - k} = P'(a)$

B) $P'(a)(k - m) = P(k) - P(m)$

C) $\dfrac{m - k}{P(m) - P(k)} = a$

D) $\dfrac{m - k}{P(m) - P(k)} = P'(a)$

E) None of these

822. The Mean Value Theorem does not apply to $f(x) = |x - 3|$ on $[1, 4]$ because

A) $f(x)$ is not continuous on $[1, 4]$

B) $f(x)$ is not differentiable on $(1, 4)$

C) $f(1) \neq f(4)$

D) $f(1) > f(4)$

E) None of these

823. Which of the following function fails to satisfy the conclusion of the Mean Value Theorem on the given interval?

A) $3x^{2/3} - 1;\ [1, 2]$

B) $|3x - 2|;\ [1, 2]$

C) $4x^3 - 2x + 3;\ [0, 2]$

D) $\sqrt{x - 2};\ [3, 6]$

E) None of these

Success and failure have much in common that is good. Both mean you're trying. —*Frank Tyger*

824. If a function F is differentiable on $[-4,4]$, then which of the following statements is true?

A) F is not continuous on $[-5,5]$

B) F is not differentiable on $[-5,5]$

C) $F'(c)=0$ for some c in the interval $[-4,4]$

D) The conclusion of the Mean Value Theorem applies to F

E) None of these

825. The function $G(x)=\dfrac{(x-2)(x-3)}{x-1}$ does not satisfy the hypothesis of Rolle's Theorem on the interval $[-3,2]$ because

A) $G(-3)=G(2)=0$

B) $G(x)$ is not differentiable on $[-3,2]$

C) $G(x)$ is not continuous on $[-3,2]$

D) $G(0)\neq 0$

E) None of these

826. The function F below satisfies the conclusion of Rolle's Theorem in the interval $[a,b]$ because

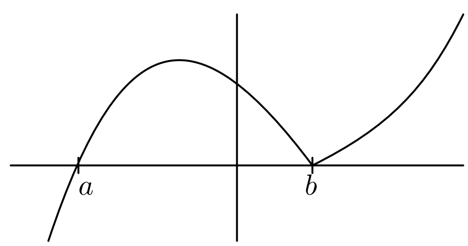
A) F is continuous on $[a,b]$

B) F is differentiable on (a,b)

C) $F(a)=F(b)=0$

D) All three statements A, B and C

E) None of these



827. The intervals for which the function $F(x)=x^4-4x^3+4x^2+6$ increases are

A) $x<0$, $1<x<2$

B) only $x>2$

C) $0<x<1$, $x>2$

D) only $0<x<1$

E) only $1<x<2$

828. If $Q(x)=(3x+2)^3$, then the third derivative of Q at $x=0$ is

A) 0 B) 9 C) 54 D) 162 E) 224

829. The function $M(x) = x^4 - 4x^2$ has

A) one relative minimum and two relative maxima

B) one relative minimum and one relative maximum

C) no relative minima and two relative maxima

D) two relative minima and no relative maxima

E) two relative minima and one relative maximum

830. The total number of all relative extrema of the function F whose derivative is $F'(x) = x(x-3)^2(x-1)^4$ is

A) 0 B) 1 C) 2 D) 3 E) None of these

831. The function $F(x) = x^{2/3}$ on $[-8, 8]$ does not satisfy the conditions of the Mean Value Theorem because

A) $F(0)$ does not exist

B) F is not continuous on $[-8, 8]$

C) $F(1)$ does not exist

D) F is not defined for $x < 0$

E) $F'(0)$ does not exist

832. If c is the number defined by Rolle's Theorem, then for $R(x) = 2x^3 - 6x$ on the interval $0 \leq x \leq \sqrt{3}$, c must be

A) 1 B) -1 C) ± 1 D) 0 E) $\sqrt{3}$

833. Find the sum of the values of a and b such that $F(x) = 2ax^2 + bx + 3$ has a relative extremum at $(1, 2)$.

A) $\frac{3}{2}$ B) $\frac{5}{2}$ C) 1 D) -1 E) None of these

834. Which of the following statements are true of the graph of $F(x)$ shown below?

I. There is a horizontal asymptote at $y = 0$.
II. There are three inflection points.
III. There are no absolute extrema.

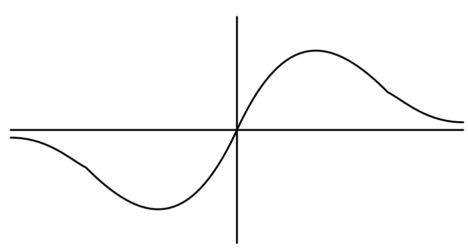
A) I only

B) I, II only

C) I, III only

D) II, III only

E) None are true



It is not enough to have a good mind. The main thing is to use it well. —*Rene Descartes*

A.P. Calculus Test Three
Section One
Multiple-Choice
Calculators Allowed

Time—45 minutes

Number of Questions—15

The scoring for this section is determined by the formula

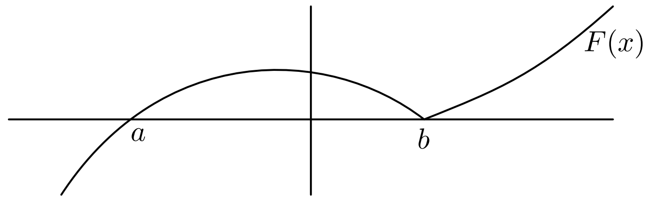
$$[C - (0.25 \times I)] \times 1.8$$

where C is the number of correct responses and I is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:



1. The function F above satisfies the conclusion of Rolle's Theorem in the interval $[a, b]$ because

I. F is continuous.
II. F is differentiable on (a, b).
III. $F(a) = F(b) = 0$.

A) I only

B) II only

C) I and III only

D) I, II, and III

E) F does not satisfy Rolle's Theorem

---

2. If $Q(x) = (3x + 2)^3$, then the third derivative of Q at $x = 0$ is

A) 0

B) 9

C) 54

D) 162

E) 224

---

3. If a function g is differentiable on the interval $[-4, 4]$, then which of the following statements is true?

A) g is not continuous on $[-5, 5]$.

B) g is not differentiable on $[-5, 5]$.

C) $g'(c) = 0$ for some c in $[-4, 4]$.

D) The conclusion of the Mean Value Theorem applies to g.

E) None of the above statements are true.

4. The value of c guaranteed to exist by the Mean Value Theorem for $f(x) = x^2$ in the interval $[0, 3]$ is

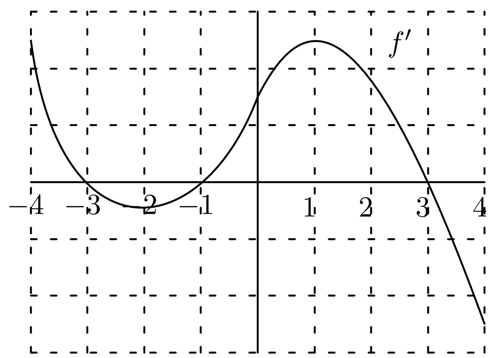
A) 1

B) 2

C) $\frac{3}{2}$

D) $\frac{1}{2}$

E) None of these



5. The graph of the derivative of a function f is shown above. Which of the following are true about the original function f ?

I. f is increasing on the interval $(-2, 1)$.
II. f is continuous at $x = 0$.
III. f has an inflection point at $x = -2$.

A) I only

B) II only

C) III only

D) II and III only

E) I, II, and III

---

6. Two particles move along the x-axis and their positions at time $0 \leq t \leq 2\pi$ are given by $x_1 = \cos t$ and $x_2 = e^{(t-3)/2} - 0.75$. For how many values of t do the two particles have the same velocity?

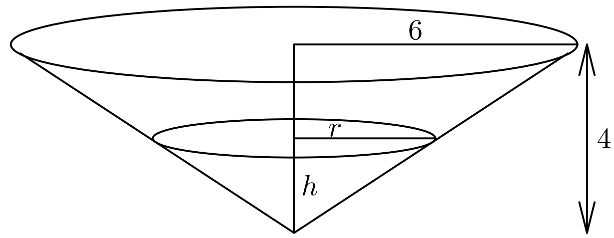
A) 0

B) 1

C) 2

D) 3

E) 4



7. The conical reservoir shown above has diameter 12 feet and height 4 feet. Water is flowing into the reservoir at the constant rate of 10 cubic feet per minute. At the instant when the surface of the water is 2 feet above the vertex, the water level is rising at the rate of

A) 0.177 ft per min

B) 0.354 ft per min

C) 0.531 ft per min

D) 0.708 ft per min

E) 0.885 ft per min

---

8. The position of a particle moving on the x-axis, starting at time $t = 0$, is given by $x(t) = (t - a)^3(t - b)$, where $0 < a < b$. Which of the following statements are true?

I. The particle is at a positive position on the x-axis at time $t = \frac{a+b}{2}$.
II. The particle is at rest at time $t = a$.
III. The particle is moving to the right at time $t = b$.

A) I only

B) II only

C) III only

D) I and II only

E) II and III only

---

9. Let the function f be differentiable on the interval $[0, 2.5]$ and define g by $g(x) = f(f(x))$. Use the table below to estimate $g'(1)$.

x	0.0	0.5	1.0	1.5	2.0	2.5
$f(x)$	1.7	1.8	2.0	2.4	3.1	4.4

A) 0.8

B) 1.2

C) 1.6

D) 2.0

E) 2.4

10. Which of the following are true about a particle that starts at $t = 0$ and moves along a number line if its position at time t is given by $s(t) = (t-2)^3(t-6)$?

I. The particle is moving to the right for $t > 5$.
II. The particle is at rest at $t = 2$ and $t = 6$.
III. The particle changes direction at $t = 2$.

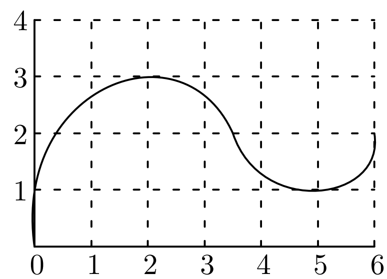
A) I only

B) II only

C) III only

D) I and III only

E) None are true.



Graph of f

11. The graph of the function f is shown above. Which of the following statements are true?

I. $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h} = f'(5)$.
II. $\dfrac{f(5)-f(2)}{5-2} = \dfrac{2}{3}$.
III. $f''(1) \le f''(5)$.

A) I and II only

B) I and III only

C) II and III only

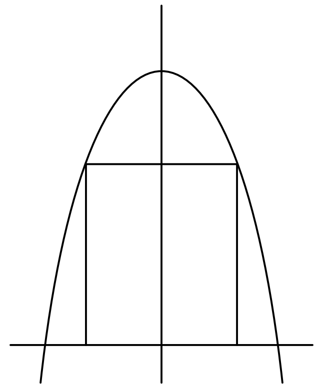
D) I, II, and III

E) None of these

---

12. If $x^2 - y^2 = 25$, then $\dfrac{d^2y}{dx^2} =$

A) $-\dfrac{x}{y}$ B) $\dfrac{5}{y^2}$ C) $-\dfrac{x^2}{y^3}$ D) $-\dfrac{25}{y^3}$ E) $\dfrac{4}{y^3}$



13. A rectangle with one side on the x-axis has its upper vertices on the graph of $y = 4 - x^2$, as shown in the figure above. What is the maximum area of the rectangle?

A) 1.155

B) 1.855

C) 3.709

D) 6.158

E) 12.316

---

14. Let f be a twice-differentiable function of x such that, when $x = c$, f is decreasing, concave up, and has an x-intercept. Which of the following is true?

A) $f(c) < f'(c) < f''(c)$

B) $f(c) < f''(c) < f'(c)$

C) $f'(c) < f(c) < f''(c)$

D) $f'(c) < f''(c) < f(c)$

E) $f''(c) < f(c) < f'(c)$

---

15. If $f'(x) = \arctan(x^3 - x)$, at how many points is the tangent line to the graph of $f(x)$ parallel to the line $y = 2x$?

A) None

B) 1

C) 2

D) 3

E) Infinitely many

A.P. Calculus Test Three
Section Two
Free-Response
No Calculators

Time—45 minutes

Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $y'(2) = 3$ may not be written as `nDeriv(Y1,X,2)=3`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. A particle moves along a line so that at any time t its position is given by $x(t) = 2\pi t + \cos 2\pi t$.

a) Find the velocity at time t.

b) Find the acceleration at time t.

c) What are all values of t, for $0 \le t \le 3$, for which the particle is at rest?

d) What is the maximum velocity?

---

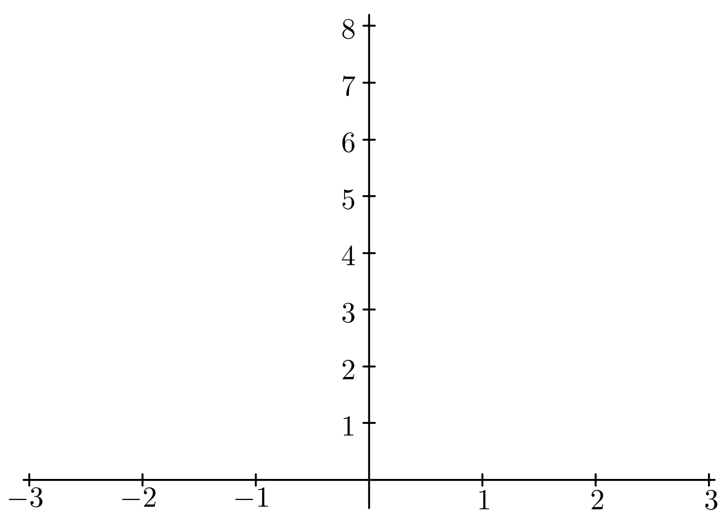
2. A function f is continuous on the closed interval $[-3, 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The function f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	positive	fails to exist	negative	0	negative
$f''(x)$	positive	fails to exist	positive	0	negative

a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.

b) What are the x-coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.

c) On the axes provided, sketch a graph that satisfies the given properties of f.



---

3. Let f be the function given by $f(x) = x^3 - 5x^2 + 3x + k$, where k is a constant.

a) On what intervals is f increasing?

b) On what intervals is the graph of f concave downward?

c) Find the value of k for which f has 11 as its relative minimum.

CHAPTER 4

INTEGRALS

4.1 The ANTIderivative!

835. For each part of this problem you are given two functions, f and g. Differentiate both functions. How are the derivatives related? How are f and g related? Is it possible for different functions to have the same derivative? What must be true of such functions?

a) $f(x) = (x-1)^3$ and $g(x) = x^3 - 3x^2 + 3x$

b) $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$

836. Let f and g be two differentiable functions such that $f'(x) = g'(x)$ for all x. What additional condition from the choices below is necessary in order to conclude that $f(x) = g(x)$ for all values of x?

A) $f''(x) = g''(x)$ for all x

B) $f(0) = g(0)$

C) f and g are continuous

D) No additional condition will allow you to conclude that $f(x) = g(x)$

E) No additional condition is required

Find antiderivatives for each of the following by considering derivative rules in reverse.

837. $6x$

838. x^7

839. $x^7 - 6x + 8$

840. $-3x^{-4}$

841. x^{-4}

842. $x^{-4} + 2x + 3$

843. $-\dfrac{2}{x^3}$

844. $\dfrac{1}{2x^3}$

845. $x^3 - \dfrac{1}{x^3}$

846. $\sec^2 x$

847. $\frac{2}{3}\sec^2\left(\frac{x}{3}\right)$

848. $-\sec^2\left(\frac{3x}{2}\right)$

849. $\sec x \tan x$

850. $4 \sec 3x \tan 3x$

851. $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$

852. $x + 1$

853. $3t^2 + \frac{1}{2}t$

854. $\dfrac{1}{x} - \dfrac{5}{x^2+1}$

855. $\dfrac{1}{x^2} - x^2 - 3$

856. An antiderivative of $y = e^{x+e^x}$ is

A) $\dfrac{e^{x+e^x}}{1+e^x}$ B) $(1+e^x)e^{x+e^x}$ C) e^{1+e^x} D) e^{x+e^x} E) e^{e^x}

"Necessity is the mother of invention" is a silly proverb. "Necessity is the mother of futile dodges" is much nearer the truth. —*Alfred North Whitehead*

4.2 Derivative Rules Backwards

Find the following indefinite integrals.

857. $\displaystyle\int (x^3+2)\ dx$

858. $\displaystyle\int (x^2-2x+3)\ dx$

859. $\displaystyle\int (x^{3/2}+2x+1)\ dx$

860. $\displaystyle\int \left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)\ dx$

861. $\displaystyle\int \sqrt[3]{x^2}\ dx$

862. $\displaystyle\int \frac{1}{x^3}\ dx$

863. $\displaystyle\int \frac{x^2+1}{x^2}\ dx$

864. $\displaystyle\int x^2\sqrt{x}\ dx$

865. $\displaystyle\int 3\ dx$

866. $\displaystyle\int (x^2-\sin x)\ dx$

867. $\displaystyle\int (1-\csc x\cot x)\ dx$

868. $\displaystyle\int (\sec^2\theta-\sin\theta)\ d\theta$

869. $\displaystyle\int \sec\theta(\tan\theta-\sec\theta)\ d\theta$

870. $\displaystyle\int \frac{8}{x^{3/5}}\ dx$

871. $\displaystyle\int \frac{-3x}{\sqrt[3]{x^4}}\ dx$

872. $\displaystyle\int 7x^3(3x^4-2x)\ dx$

873. $\displaystyle\int \frac{7\sqrt{x}-3x^2-3}{4\sqrt{x}}\ dx$

874. $\displaystyle\int e^x\ dx$

875. $\displaystyle\int 2^x\ln 2\ dx$

876. $\displaystyle\int 5e^x\ dx$

877. $\displaystyle\int \frac{1}{x^2+1}\ dx$

878. $\displaystyle\int \frac{3}{\sqrt{1-x^2}}\ dx$

The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. Of course I do not here speak of that beauty that strikes the senses, the beauty of qualities and appearances; not that I undervalue such beauty, far from it, but it has nothing to do with science; I mean that profounder beauty which comes from the harmonious order of the parts, and which a pure intelligence can grasp. —*Henri Poincaré*

4.3 The Method of Substitution

Find the following indefinite integrals.

879. $\displaystyle\int -2x\sqrt{9-x^2}\ dx$

880. $\displaystyle\int x(4x^2+3)^3\ dx$

881. $\displaystyle\int \frac{x^2}{(1+x^3)^2}\ dx$

882. $\displaystyle\int \left(x^2+\frac{1}{9x^2}\right)\ dx$

883. $\displaystyle\int \frac{x^2+3x+7}{\sqrt{x}}\ dx$

884. $\displaystyle\int \left(\frac{t^3}{3}+\frac{1}{4t^2}\right)\ dt$

885. $\displaystyle\int \sin 2x\ dx$

886. $\displaystyle\int \cos 6x\ dx$

887. $\displaystyle\int \tan^4\theta\sec^2\theta\ d\theta$

888. $\displaystyle\int \frac{\sin\theta}{\cos^2\theta}\ d\theta$

889. $\displaystyle\int \cos\frac{\theta}{2}\ d\theta$

890. $\displaystyle\int x\sqrt{2x+1}\ dx$

891. $\displaystyle\int x^2\sqrt{1-x}\ dx$

892. $\displaystyle\int \sqrt{4x-3}\ dx$

893. $\displaystyle\int x^4\sqrt{3x^5-4}\ dx$

894. $\displaystyle\int \frac{3x^6}{(2x^7-1)^5}\ dx$

895. $\displaystyle\int 4x\sqrt{5x-2}\ dx$

896. $\displaystyle\int 12x^2\sin(4x^3)\ dx$

897. $\displaystyle\int 4e^x\cos(4e^x)\ dx$

898. $\displaystyle\int 3^{3t}\ln 3\ dt$

899. $\displaystyle\int 6^{2x^2-3}x\ln 6\ dx$

900. $\displaystyle\int 2^{5x}\ dx$

901. $\displaystyle\int \frac{1}{\sqrt{5x+4}}\ dx$

902. $\displaystyle\int 3y\sqrt{7-3y^2}\ dy$

903. $\displaystyle\int \cos(3z+4)\ dz$

904. $\displaystyle\int \frac{1}{t^2}e^{1/t}\ dt$

905. $\displaystyle\int \sec(x+\tfrac{\pi}{2})\tan(x+\tfrac{\pi}{2})\ dx$

906. $\displaystyle\int -\csc^2\theta\sqrt{\cot\theta}\ d\theta$

907. $\displaystyle\int \frac{x}{x^2+4}\ dx$

908. $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\ dx$

909. $\displaystyle\int \frac{e^x}{1+e^{2x}}\ dx$

910. $\displaystyle\int \frac{1}{x}\ dx$

The science of pure mathematics... may claim to be the most original creation of the human spirit. —*Alfred North Whitehead*

4.4 Using Geometry for Definite Integrals

Graph the integrands and use geometry to evaluate the definite integrals.

911. $\displaystyle\int_{-2}^{4} \left(\frac{x}{2}+3\right)\,dx$

912. $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$

913. $\displaystyle\int_{-2}^{1} |x|\,dx$

914. $\displaystyle\int_{-1}^{1} (2-|x|)\,dx$

915. $\displaystyle\int_{0}^{b} x\,dx$ where $b>0$

916. $\displaystyle\int_{a}^{b} 2x\,dx$ where $0<a<b$

917. Suppose f and g are continuous and that

$$\int_1^2 f(x)\,dx=-4, \qquad \int_1^5 f(x)\,dx=6, \qquad \int_1^5 g(x)\,dx=8.$$

Evaluate the following definite integrals.

a) $\displaystyle\int_2^2 g(x)\,dx$

b) $\displaystyle\int_5^1 g(x)\,dx$

c) $\displaystyle\int_1^2 3f(x)\,dx$

d) $\displaystyle\int_2^5 f(x)\,dx$

e) $\displaystyle\int_1^5 [f(x)-g(x)]\,dx$

f) $\displaystyle\int_1^5 [4f(x)-g(x)]\,dx$

918. Suppose that $\displaystyle\int_{-3}^{0} g(t)\,dt=\sqrt{2}$. Find the following.

a) $\displaystyle\int_0^{-3} g(t)\,dt$

b) $\displaystyle\int_{-3}^{0} g(u)\,du$

c) $\displaystyle\int_{-3}^{0} -g(x)\,dx$

d) $\displaystyle\int_{-3}^{0} \frac{g(\theta)}{\sqrt{2}}\,d\theta$

919. A particle moves along the x-axis so that at any time $t\geq 0$ its acceleration is given by $a(t)=18-2t$. At time $t=1$ the velocity of the particle is 36 meters per second and its position is $x=21$.

a) Find the velocity function and the position function for $t\geq 0$.

b) What is the position of the particle when it is farthest to the right?

When you feel how depressingly
Slowly you climb,
It's well to remember
That things take time.

—*Piet Hein*

4.5 Some Riemann Sums

920. The table shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the distance traveled by the engine using 10 subintervals of length 1 with a) left-hand values and b) right-hand values.

Time (seconds)	0	1	2	3	4	5	6	7	8	9	10
Velocity (in/sec)	0	12	22	10	5	13	11	6	2	6	0

921. The table shows the velocity of a vintage sports car accelerating from 0 to 142 miles per hour in 36 seconds (0.01 hours).

hours	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010
mph	0	40	62	82	96	108	116	125	132	137	142

a) Use a Riemann sum to estimate how far the car traveled during the 36 seconds it took to reach 142 mph.

b) Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

922. Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (hours)	0	1	2	3	4	5	6	7	8
Leakage (gal./hour)	50	70	97	136	190	265	369	516	720

a) Give an upper and lower estimate of the total quantity of oil that has escaped after 5 hours.

b) Give an upper and lower estimate of the total quantity of oil that has escaped after 8 hours.

c) The tanker continues to leak 720 gal/hr after the first 8 hours. If the tanker originally contained 25,000 gallons of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

923. A rectangular swimming pool is 30 ft wide and 50 ft long. The table below shows the depth of the water at 5 ft intervals from one end of the pool to the other. Estimate the volume of water in the pool by computing the average of the left-hand and right-hand Riemann sums.

Position (ft)	0	5	10	15	20	25	30	35	40	45	50
Depth (ft)	6.0	8.2	9.1	9.9	10.5	11.0	11.5	11.9	12.3	12.7	13.0

Don't confuse being busy with accomplishment. —*Anonymous*

4.6 The MVT and the FTC

Find y', the derivative of the function y, for each of the following using the Fundamental Theorem of Calculus.

924. $y = \int_0^x (t+2)\ dt$

925. $y = \int_8^x \sqrt[3]{t}\ dt$

926. $y = \int_{\pi/4}^x \sec^2 t\ dt$

927. $y = \int_{-2}^x (t^2 - 2t)\ dt$

928. $y = \int_{-1}^x \sqrt{t^4+1}\ dt$

929. $y = \int_0^x \tan^4 t\ dt$

930. $y = \int_0^x t\cos t\ dt$

931. $y = \int_1^x \frac{t^2}{1+t^2}\ dt$

932. $y = \int_x^{x+2} (4t+1)\ dt$

933. $y = \int_0^{\sin x} \sqrt{t}\ dt$

934. $y = \int_0^{x^3} \sin(t^2)\ dt$

935. $y = \int_0^{3x} \sqrt{1+t^3}\ dt$

Find the average value of each of the following functions on the given interval.

936. $f(x) = x - 2\sqrt{x};\ \ [0,2]$

937. $f(x) = \frac{9}{x^3};\ \ [1,3]$

938. $f(x) = 2\sec^2 x;\ \ [-\frac{\pi}{4}, \frac{\pi}{4}]$

939. $f(x) = \cos x;\ \ [-\frac{\pi}{3}, \frac{\pi}{3}]$

Find exact values for each of the following definite integrals.

940. $\int_0^1 (x^2 + \sqrt{x})\ dx$

941. $\int_0^{\pi/3} 2\sec^2 x\ dx$

942. $\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y)\ dy$

943. $\int_4^9 \frac{1-\sqrt{u}}{\sqrt{u}}\ du$

944. $\int_2^7 3\ dx$

945. $\int_{-1}^8 (x^{1/3} - x)\ dx$

946. $\int_{-1}^1 (t^2 - 2)\ dt$

947. $\int_0^3 (3x^2 + x - 2)\ dx$

948. $\int_1^2 \left(\frac{3}{x^2} - 1\right)\ dx$

949. $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right)\ du$

950. $\int_{-\pi/3}^{\pi/3} 4\sec\theta\tan\theta\ d\theta$

951. $\int_0^2 3^x \ln 3\ dx$

952. $\int_0^{\ln 5} e^x\ dx$

953. $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\ dx$

4.7 The FTC, Graphically

954. Use the function f in the figure below and the function g defined by $g(x) = \int_0^x f(t)\,dt$.

a) Complete the table.

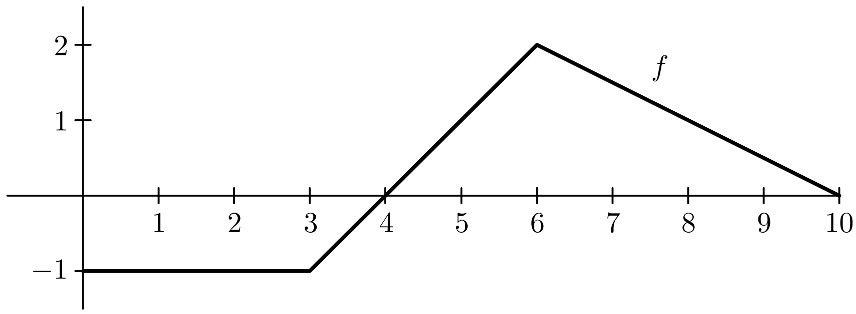
x	0	1	2	3	4	5	6	7	8	9	10
$g(x)$											

b) Plot the points from the table in part (a).

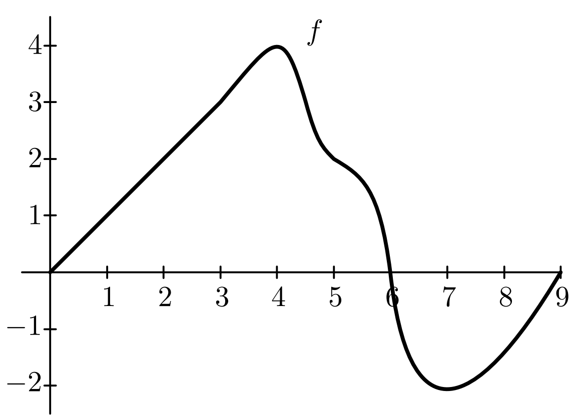
c) Where does g have its minimum? Explain.

d) Which four consecutive points are collinear? Explain.

e) Between which two consecutive points does g increase at the greatest rate? Explain.



955. Suppose f is the differentiable function shown in the accompanying graph and that the position at time t (in seconds) of a particle moving along the coordinate axis is $s(t) = \int_0^t f(x)\,dx$ meters. Use the graph to answer the following questions. Justify your answers.



a) What is the particle's velocity at time $t = 5$?

b) Is the acceleration of the particle at time $t = 5$ positive or negative?

c) What is the particle's position at $t = 3$?

d) At what time during the first 9 seconds does s have its largest value?

e) Approximately when is the acceleration zero?

f) When is the particle moving toward the origin? Away from the origin?

g) On which side of the origin does the particle lie at time $t = 9$?

Black holes are where God divided by zero. —*Steven Wright*

4.8 Definite and Indefinite Integrals

FIND THE FOLLOWING INDEFINITE INTEGRALS.

956. $\displaystyle\int (x^2-1)^2\ dx$

957. $\displaystyle\int \tfrac{1}{2}\cos 5x\ dx$

958. $\displaystyle\int 2^{5w}\ dw$

959. $\displaystyle\int \sin(5\theta)\cos(5\theta)\ d\theta$

960. $\displaystyle\int \frac{4x}{(4x^2-1)^5}\ dx$

961. $\displaystyle\int \frac{z^3-2z^2-5}{z^2}\ dz$

962. $\displaystyle\int (x^2+14x+49)^{35}\ dx$

963. $\displaystyle\int e^x(e^x-1)^7\ dx$

964. $\displaystyle\int [\sin(5\theta)+1]^4\cos(5\theta)\ d\theta$

965. $\displaystyle\int 2^{\log_2 7x}\ dx$

FIND EXACT VALUES FOR THE FOLLOWING DEFINITE INTEGRALS.

966. $\displaystyle\int_{-1}^{1} x(x^2+1)^3\ dx$

967. $\displaystyle\int_{0}^{1} x\sqrt{1-x^2}\ dx$

968. $\displaystyle\int_{0}^{4} \frac{1}{\sqrt{2x+1}}\ dx$

969. $\displaystyle\int_{0}^{2} \frac{x}{\sqrt{1+2x^2}}\ dx$

970. $\displaystyle\int_{1}^{9} \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\ dx$

971. $\displaystyle\int_{0}^{2} x\sqrt[3]{x^2+4}\ dx$

972. $\displaystyle\int_{1}^{2} (x-1)\sqrt{2-x}\ dx$

973. $\displaystyle\int_{0}^{4} \frac{x}{\sqrt{2x+1}}\ dx$

974. $\displaystyle\int_{0}^{\pi/2} \cos(\tfrac{2x}{3})\ dx$

975. $\displaystyle\int_{\pi/3}^{\pi/2} (x+\cos x)\ dx$

976. $\displaystyle\int_{0}^{7} x\sqrt[3]{x+1}\ dx$

977. $\displaystyle\int_{-2}^{6} x^2\sqrt[3]{x+2}\ dx$

FIND THE AREA UNDER THE CURVE OVER THE GIVEN INTERVAL.

978. $y=2\sin x+\sin(2x);\quad [0,\pi]$

979. $y=\sin x+\cos(2x);\quad [0,\pi]$

980. $y=\sec^2(\frac{x}{2});\quad [\frac{\pi}{2},\frac{2\pi}{3}]$

981. $y=\csc(2x)\cot(2x);\quad [\frac{\pi}{12},\frac{\pi}{4}]$

No one really understood music unless he was a scientist, her father had declared, and not just any scientist, either, oh, no, only the real ones, the theoreticians, whose language is mathematics. She had not understood mathematics until he had explained to her that it was the symbolic language of relationships. "And relationships," he had told her, "contained the essential meaning of life." —*Pearl S. Buck,* The Goddess Abides, Part 1

4.9 Integrals Involving Logarithms and Exponentials

Find the following indefinite integrals.

982. $\displaystyle\int \frac{1}{x+1}\,dx$

983. $\displaystyle\int \frac{x}{x^2+1}\,dx$

984. $\displaystyle\int \frac{x^2-4}{x}\,dx$

985. $\displaystyle\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$

986. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

987. $\displaystyle\int \frac{1}{\sqrt{x+1}}\,dx$

988. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$

989. $\displaystyle\int \frac{2x}{(x-1)^2}\,dx$

990. $\displaystyle\int \frac{\cos\theta}{\sin\theta}\,d\theta$

991. $\displaystyle\int \csc(2\theta)\,d\theta$

992. $\displaystyle\int \frac{\cos\theta}{1+\sin\theta}\,d\theta$

993. $\displaystyle\int \frac{\sec\theta\tan\theta}{\sec\theta-1}\,d\theta$

994. $\displaystyle\int 5e^{5x}\,dx$

995. $\displaystyle\int \frac{e^{-x}}{1+e^{-x}}\,dx$

996. $\displaystyle\int e^x\sqrt{1-e^x}\,dx$

997. $\displaystyle\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$

998. $\displaystyle\int \frac{5-e^x}{e^{2x}}\,dx$

999. $\displaystyle\int e^{\sin(\pi x)}\cos(\pi x)\,dx$

1000. $\displaystyle\int e^{-x}\tan(e^{-x})\,dx$

1001. $\displaystyle\int 3^x\,dx$

1002. $\displaystyle\int 5^{-x^2}x\,dx$

1003. $\displaystyle\int \frac{3^{2x}}{1+3^{2x}}\,dx$

Find exact values for each of the following definite integrals.

1004. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx$

1005. $\displaystyle\int_{-1}^1 \frac{1}{x+2}\,dx$

1006. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

1007. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx$

1008. $\displaystyle\int_\pi^{2\pi} \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

1009. $\displaystyle\int_1^5 \frac{x+5}{x}\,dx$

1010. $\displaystyle\int_0^1 e^{-2x}\,dx$

1011. $\displaystyle\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

1012. $\displaystyle\int_{-1}^2 2^x\,dx$

1013. $\displaystyle\int_0^1 \frac{3^{4x}(4\ln 3)}{3^{4x}+1}\,dx$

4.10 It Wouldn't Be Called the Fundamental Theorem If It Wasn't Fundamental

In the following four problems, find $F'(x)$.

1014. $F(x) = \int_1^x \frac{1}{t}\,dt$

1015. $F(x) = \int_0^x \tan t\,dt$

1016. $F(x) = \int_x^{3x} \frac{1}{t}\,dt$

1017. $F(x) = \int_1^{x^2} \frac{1}{t}\,dt$

1018. Let f be a continuous function with an antiderivative F on the interval $[a, b]$. Let c be any point in the interval. State whether the following are true or false. If false, then correct the statement or give an example to show why it is false.

a) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

b) $\int_a^b F(x)\,dx = f(b) - f(a)$

c) $\int_a^b f(x)\,dx \geq 0$

d) $\int_a^b cf(x)\,dx = c(F(b) - F(a))$

e) $\int_a^b f(x)\,dx = f(m)(b - a)$ for some m in $[a, b]$

1019. *An Average Value Investigation, Part 1*

a) Find the average values of $f(x) = x$, $f(x) = x^2$, and $f(x) = x^3$ over the interval $[0, 1]$.

b) From the pattern established in part (a), what is the average value of $f(x) = x^n$, for an integer $n \geq 1$?

c) What does the answer to part (b) imply about the average value of $f(x) = x^n$, as n gets larger and larger? Can you explain this from the graph of $f(x) = x^n$?

1020. *An Average Value Investigation, Part 2*

a) Find the average values of $f(x) = x$, $f(x) = x^{1/2}$, and $f(x) = x^{1/3}$ over the interval $[0, 1]$.

b) From the pattern established in part (a), what is the average value of $f(x) = x^{1/n}$, for an integer $n \geq 1$?

c) What does the answer to part (b) imply about the average value of $f(x) = x^{1/n}$, as n gets larger and larger? Can you explain this from the graph of $f(x) = x^{1/n}$?

The most practical, beautiful philosophy in the world won't work–if you won't. —*Will Rogers*

1021. Find the average value of the following.

a) $f(x) = x - 2$ on $[1, 3]$

b) $f(x) = x^3 - x$ on $[-1, 1]$

c) $f(x) = \cos x$ on $[0, \pi]$

d) What is the relationship between the graphs and intervals that make these so easy?

1022 (AP). Suppose that $5x^3 + 40 = \int_c^x f(t)\,dt$.

a) What is $f(x)$?

b) Find the value of c.

1023. Let $G(x) = \int_0^x \sqrt{16 - t^2}\,dt$.

a) Find $G(0)$.

b) Does $G(2) = G(-2)$? Does $G(2) = -G(-2)$?

c) What is $G'(2)$?

d) What are $G(4)$ and $G(-4)$?

1024. Marcus is caught speeding. The fine is \$3.00 per minute for each mile per hour above the speed limit. Since he was clocked at speeds as much as 64 mph over a 6-minute period, the judge fines him:

$$(\$3.00)(\text{number of minutes})(\text{mph over } 55) = (\$3.00)(6)(64-55) = \$162.00$$

Marcus believes that the fine is too large since he was going 55 mph at times $t = 0$ and $t = 6$ minutes, and was going 64 mph only at $t = 3$. He reckons, in fact, that his speed v is given by $v = 55 + 6t - t^2$.

a) Show that Marcus's equation does give the correct speed at times $t = 0$, $t = 3$ and $t = 6$.

b) Marcus argues that since his speed varied, the fine should be determined by calculus rather than by arithmetic. What should he propose to the judge as a reasonable fine?

1025. If $F(x) = \int_0^3 t\sqrt{t+9}\,dt$, then $F'(1) = 0$. Why?

1026. Evaluate $\frac{d}{dx}\int_a^b x^3\,dx$ where a and b are real numbers.

1027. If $g(x) = \int_0^x \sqrt{u^2+2}\,du$, what is $\frac{d^2g}{dx^2}$?

1028. If $g(x) = \int_0^{x^2} f(u)\,du$, what is $\frac{dg}{dx}$?

I have no special gift; I am only passionately curious. —*Albert Einstein*

4.11 Definite and Indefinite Integrals Part 2

Find exact values for the following definite integrals.

1029. $\displaystyle\int_{-2}^{4} 10\ dx$

1030. $\displaystyle\int_{4}^{-2} dx$

1031. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\ dx$

1032. $\displaystyle\int_{-1}^{1} |x|\ dx$

1033. $\displaystyle\int_{0}^{2} |2x-3|\ dx$

1034. $\displaystyle\int_{0}^{3\pi/2} |\sin x|\ dx$

1035. $\displaystyle\int_{-\pi/2}^{\pi} |\cos x|\ dx$

1036. $\displaystyle\int_{0}^{3} |x^2-4|\ dx$

1037. $\displaystyle\int_{-2}^{2} (5-|x|)\ dx$

1038. $\displaystyle\int_{0}^{4} f(x)\ dx$ where $f(x) = \begin{cases} 2 & 0 \le x < 1 \\ 5 & 1 \le x < \frac{3}{2} \\ 1 & \frac{3}{2} \le x < 4 \\ 5 & x = 4 \end{cases}$

1039. $\displaystyle\int_{0}^{10} f(x)\ dx$ where $f(x) = \begin{cases} 2x & 0 \le x < 4 \\ 3 & 4 \le x < 6 \\ 2x & 6 \le x < 10 \end{cases}$

1040. $\displaystyle\int_{1/2}^{5} f(x)\ dx$ where $f(x) = \begin{cases} \frac{1}{x} & \frac{1}{2} \le x \le 2 \\ x & 2 < x \le 5 \end{cases}$

1041. $\displaystyle\int_{0}^{9} f(x)\ dx$ where $f(x) = \begin{cases} x^2 & 0 < x < 2 \\ 4 & 2 \le x < 5 \\ 9-x & 5 \le x < 9 \end{cases}$

1042. $\displaystyle\int_{0}^{5} g(x)\ dx$ where $g(x) = \begin{cases} 2x^3-5x^2+3 & 0 < x < 2 \\ 10+x & 2 \le x < 3 \\ 20-x & 3 \le x < 5 \end{cases}$

1043. $\displaystyle\int_{0}^{2} f(x)\ dx$ where $f(x) = \begin{cases} x^3 & 0 \le x < 1 \\ 2-x & 1 < x \le 2 \end{cases}$

1044. Find a curve $y = f(x)$ with the following properties:

I. $\dfrac{d^2y}{dx^2} = 6x$

II. Its graph passes through $(0, 1)$

III. Its graph has a horizontal tangent at $(0, 1)$

In science one tries to tell people, in such a way as to be understood by everyone, something that no one ever knew before. But in poetry, it's the exact opposite. —*Paul Dirac*

4.12 Regarding Riemann Sums

1045. Let $f(x) = x^2 + x$. Consider the region bounded by the graph of f, the x-axis, and the line $x = 2$. Divide the interval $[0, 2]$ into 8 equal subintervals. Draw a picture to help answer the following.

a) Obtain a lower estimate for the area of the region by using the left-hand endpoint of each subinterval.

b) Obtain an upper estimate for the area of the region by using the right-hand endpoint of each subinterval.

c) Find an approximation for the area that is better than either of the answers obtained in parts (a) and (b).

d) Without calculating the exact area, determine whether the answer in part (c) is larger or smaller than the exact area. Justify your answer.

1046. Let $f(x) = 4 - x^2$. Repeat problem 1045 with this new function function f.

1047. In order to determine the average temperature for the day, meteorologist Sam Anthuh Alun decides to record the temperature at eight times during the day. She further decides that these recordings do not have to be equally spaced during the day because she does not need to make several readings during those periods when the temperature is not changing much (as well as not wanting to get up in the middle of the night). She decides to make one reading at some time during each of the intervals in the table below.

Time	12AM-5AM	5AM-7AM	7AM-9AM	9AM-1PM
Temp	42°	57°	72°	84°

Time	1PM-4PM	4PM-7PM	7PM-9PM	9PM-12AM
Temp	89°	75°	66°	52°

a) Using Riemann sums, write a formula for the average temperature for this day.

b) Calculate the average temperature.

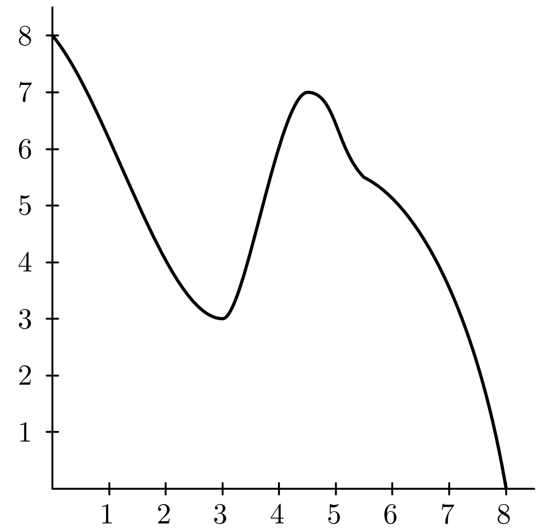
1048. Assume the following function f is a decreasing function on the interval $[0, 4]$ and that the following is a table showing some function values.

x	0	1	1.5	3	4
$f(x)$	4	3	2	1.5	1

Employ a Riemann sum to approximate $\int_0^4 f(x)\,dx$. Use a method so that your approximation will either be less than the value of the definite integral or will be greater than the definite integral. Finally, indicate whether your approximation is less than or greater than the value of the definite integral.

Nature laughs at the difficulties of integration. —*Pierre Simon Laplace*

1049. Let f be the function graphed at the right. Which of the following is the best estimate of $\int_1^6 f(x)\,dx$? Justify your answer.



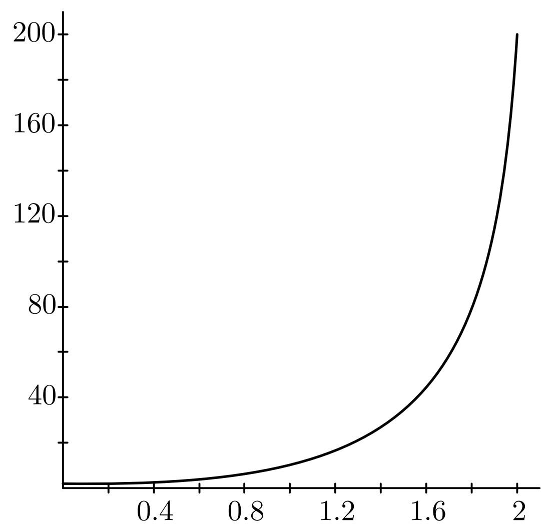
A) -24

B) 9

C) 26

D) 38

1050. The graph of a function f is given in the figure at right. When asked to estimate $\int_1^2 f(x)\,dx$ to five decimal place accuracy, a group of Georgia Southern University calculus students submitted the following answers.



A) -4.57440

B) 4.57440

C) 45.74402

D) 457.44021

Although one of these responses is correct, the other three are "obviously" incorrect. Using arguments Georgia Southern students would understand, identify the correct answer and explain why each of the others cannot be correct.

1051. Consider the following table of values of a continuous function f at different values of x.

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	0.14	0.21	0.28	0.36	0.44	0.54	0.61	0.70	0.78	0.85

a) From the data given, find two estimates of $\int_1^{10} f(x)\,dx$.

b) Obtain a different estimate for the integral by taking an average value of f over each subinterval.

c) Do you think that your estimates are too big or too little? Explain.

When we ask advice, we are usually looking for an accomplice. —*Joseph-Louis Lagrange*

4.13 Definitely Exciting Definite Integrals!

1052. Let f be a continuous function on the interval $[a, b]$. State whether the following are true or false. If false, then correct the statement or give an example to show why it is false.

a) $\dfrac{d}{dx}\displaystyle\int_a^b f(x)\,dx = f'(b) - f'(a)$

b) $\displaystyle\int_a^a f(x)\,dx = 0$

c) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

1053. Let $F(x)$ be a continuous function on $[a, f]$, where $a < b < c < d < e < f$, and

$$\int_a^c F(x)\,dx = 8, \qquad \int_c^e F(x)\,dx = 5, \qquad \int_e^f F(x)\,dx = -3,$$

$$\int_b^c F(x)\,dx = 2, \qquad \int_d^e F(x)\,dx = 1.$$

Evaluate the definite integrals below.

a) $\displaystyle\int_b^e F(x)\,dx$

b) $\displaystyle\int_d^e F(x)\,dx$

c) $\displaystyle\int_a^f F(x)\,dx$

d) $\displaystyle\int_b^d F(x)\,dx$

e) $\displaystyle\int_b^a F(x)\,dx$

f) $\displaystyle\int_d^c F(x)\,dx$

g) $\displaystyle\int_a^e 3F(x)\,dx$

h) $\displaystyle\int_c^f -F(x)\,dx$

i) $\displaystyle\int_a^b \frac{F(x)}{3}\,dx + \int_d^f 5F(x)\,dx$

j) $\displaystyle\int_f^d F(x)\,dx - \int_f^b 4F(x)\,dx$

1054. Suppose that f has a positive derivative for all x and that $f(1) = 0$. Which of the following statements *must* be true of the function $g(x) = \displaystyle\int_0^x f(t)\,dt$? Justify your answers.

a) g is a differentiable function of x.

b) g is a continuous function of x.

c) The graph of g has a horizontal tangent at $x = 1$.

d) g has a local maximum at $x = 1$.

e) g has a local minimum at $x = 1$.

f) The graph of g has an inflection point at $x = 1$.

g) The graph of dg/dx crosses the x-axis at $x = 1$.

A wise man speaks because he has something to say; a fool because he has to say something. —*Plato*

4.14 How Do I Find the Area Under Thy Curve? Let Me Count the Ways...

In the following four problems, find the area under the curve on the interval $[a, b]$ by using
a) a right-hand Riemann sum on n equal subintervals;
b) a left-hand Riemann sum on n equal subintervals;
c) 2 trapezoids on equal subintervals;
d) Simpson's rule with 2 parabolas on equal subintervals; and
e) a definite integral.

1055. $y = 2x + 3;\ \ [0, 4];\ \ n = 4$

1056. $y = x^2 + 2;\ \ [1, 3];\ \ n = 4$

1057. $y = 9 - x^2;\ \ [0, 3];\ \ n = 6$

1058. $y = x^3 + 1;\ \ [1, 2];\ \ n = 2$

Find the exact area of the region bounded by the given curves.

1059. $y = 16 - x^2,\ y = 0,\ x = 0,\ x = -2$

1060. $y = x^3 + 4,\ y = 0,\ x = 0,\ x = 1$

1061. $y = e^{2x},\ y = 0,\ x = \ln 2,\ x = \ln 3$

1062. $y = \tan x,\ y = 0,\ x = \frac{\pi}{4}$

1063. $y = \dfrac{4}{1 + x^2},\ y = 0,\ x = 0,\ x = 1$

Find the average value of each function over the given interval.

1064. $F(x) = 2\sqrt{x - 1};\ \ [1, 2]$

1065. $G(x) = e^{-x};\ \ [0, 1]$

1066. $J(x) = x^n;\ \ [1, 2]$ for $n > 1$

1067. $W(x) = 3\cos 3x;\ \ [0, \frac{\pi}{6}]$

In the following problems, $s(t)$ is position, $v(t)$ is velocity, and $a(t)$ is acceleration. Find both the net distance and the total distance traveled by a particle with the given position, velocity, or acceleration function.

1068. $v(t) = t^2 - 5t + 6$, where $0 \le t \le 3$

1069. $s(t) = 3t^3 - t$, where $0 \le t \le 2$

1070. $a(t) = 2t - 9$, where $0 \le t \le 3$ and $v(2) = 13$

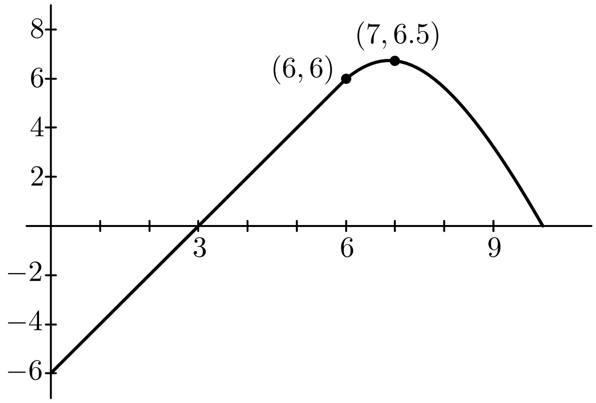
1071. $a(t) = -2t + 1$, where $0 \le t \le 3$ and $v(0) = 0$

1072. $v(t) = e^{\cos(t/2)} \sin(t/2)$, where $0 \le t \le 4\pi$

The fact is that there are few more "popular" subjects than mathematics. Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune; and there are probably more people really interested in mathematics than in music. Appearances may suggest the contrary, but there are easy explanations. Music can be used to stimulate mass emotion, while mathematics cannot; and musical incapacity is recognized (no doubt rightly) as mildly discreditable, whereas most people are so frightened of the name of mathematics that they are ready, quite unaffectedly, to exaggerate their own mathematical stupidity. —*G. H. Hardy*

4.15 Three Integral Problems

1073. Suppose that g is the differentiable function shown in the accompanying graph and that the position at time t (in seconds) of a particle moving along a coordinate axis is $s(t) = \int_0^t g(x)\,dx$ meters. Use the graph to answer the following questions. Justify your answers.



a) What is the particle's velocity at $t = 3$?

b) Is the acceleration at time $t = 3$ positive or negative?

c) What is the particle's position at $t = 3$?

d) When does the particle pass through the origin?

e) When is the acceleration zero?

f) When is the particle moving away from the origin? toward the origin?

g) On which side of the origin does the particle lie at $t = 9$?

1074. Suppose that f has a negative derivative for all x and that $f(1) = 0$. Which of the following statements must be true of the function $h(x) = \int_0^x f(t)\,dt$? Justify your answers.

a) h is a twice-differentiable function of x.

b) h and dh/dx are both continuous.

c) The graph of h has a horizontal tangent at $x = 1$.

d) h has a local maximum at $x = 1$.

e) h has a local minimum at $x = 1$.

f) The graph of h has an inflection point at $x = 1$.

g) The graph of dh/dx crosses the x-axis at $x = 1$.

1075 (Calculator). *An investigation into the accuray of the Trapezoid and Simpson's rules*

a) Using the Trapezoid rule, approximate the area between the curve $y = x \sin x$ and the x-axis from $x = 0$ to $x = \pi$, taking $n = 4, 8, 20$, and 50 subintervals.

b) Repeat part (a) using Simpson's Rule.

c) Calculate the value of the definite integral $\int_0^\pi x \sin x\,dx$ and compare it to the answers obtained in parts (a) and (b). What does this exercise suggest about the relative accuracy of the trapezoid and Simpson's rules?

Mathematicians do not study objects, but relations among objects. —*Henri Poincaré*

4.16 Trapezoid and Simpson

Approximate the value of the following definite integrals using a) the trapezoid rule and b) Simpson's rule, each with 4 subdivisions. Write out the sum, but use your calculator to do the arithmetic. Your answer must be accurate to three decimal places.

1076. $\displaystyle\int_0^8 \sqrt[3]{x}\,dx$

1077. $\displaystyle\int_1^2 \frac{1}{(x+1)^2}\,dx$

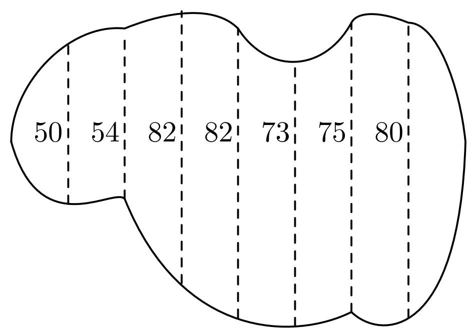
1078. $\displaystyle\int_0^1 \sqrt{x-x^2}\,dx$

1079. $\displaystyle\int_0^4 e^{-x^2}\,dx$

1080. $\displaystyle\int_0^\pi 2^{\sin x}\,dx$

1081. $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$

1082. To estimate the surface area of a pond, a surveyor takes several measurements, in feet, at 20-foot intervals, as shown in the figure. Estimate the surface area of the pond using a) the trapezoid rule and b) Simpson's rule.



1083. The table lists several measurements gathered in an experiment to approximate an unknown continuous function $y = f(x)$. Approximate the integral $\int_0^2 f(x)\,dx$ using a) the trapezoid rule and b) Simpson's rule.

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
y	4.32	4.36	4.58	5.79	6.14	7.25	7.64	8.08	8.14

1084. A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced. Use the trapezoid rule to estimate the amount of oil consumed by the generator during that week.

Day	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Oil consumption rate (liters/hour)	0.019	0.020	0.021	0.023	0.025	0.028	0.031	0.035

1085. An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every five minutes for a full hour of travel. Use the trapezoid rule to approximate the total fuel consumption; then, assuming the automobile covered 60 miles in the hour, find the fuel efficiency (in miles per gallon) for that portion of the trip.

Time	0	5	10	15	20	25	30	35	40	45	50	55	60
Gal/Hr	2.5	2.4	2.3	2.4	2.4	2.5	2.6	2.5	2.4	2.3	2.4	2.4	2.3

4.17 Properties of Integrals

1086. Suppose that f is an integrable function and that

$$\int_0^1 f(x)\,dx = 2, \qquad \int_0^2 f(x)\,dx = 1, \qquad \int_2^4 f(x)\,dx = 7.$$

a) Find $\int_0^4 f(x)\,dx$.

b) Find $\int_1^0 f(x)\,dx$.

c) Find $\int_1^2 f(x)\,dx$.

d) Explain why $f(x)$ must be negative somewhere in the interval $[1, 2]$.

e) Explain why $f(x) \geq 3.5$ for at least one value of x in the interval $[2, 4]$.

1087. Calculate the exact value of $\int_{-3}^{3} (x+5)\sqrt{9-x^2}\,dx$. *Hint:* Consider geometric methods; look at the graphs of $y = x\sqrt{9-x^2}$ and $y = \sqrt{9-x^2}$.

1088. Four calculus students disagree as to the value of the integral $\int_0^\pi \sin^8 x\,dx$. Abby says that it is equal to π. Nika says that it is equal to $35\pi/128$. Catherine claims it is equal to $3\pi/90 - 1$, while Peyton says its equal to $\pi/2$. One of them is right. Which one is it? *Hint:* Do not try to evaluate the integral; instead eliminate the three wrong answers.

1089. If you were asked to find $\int_1^2 x^2 e^{x^2}\,dx$, you could not do it analytically because you could not find an antiderivative of $x^2 e^{x^2}$. However, you should be able to estimate the size of the answer. Which is it?

A) less than 0

B) 0 to 9.999

C) 10 to 99.99

D) 100 to 999.9

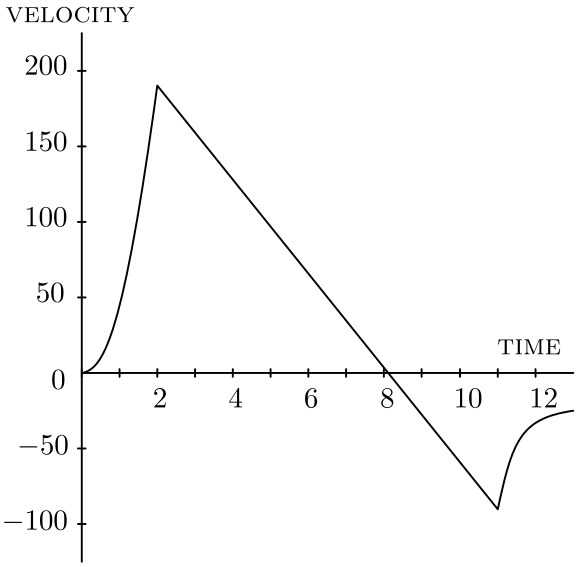
E) 1000 to 9999

F) over 10,000

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of the achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity. —*Pierre Simon Laplace*

4.18 Sample A.P. Problems on Integrals

1090. The figure shows the graph of the velocity of a model rocket for the first 12 seconds after launch.

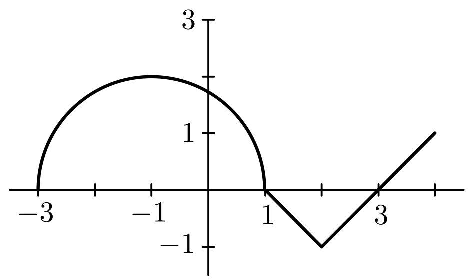


a) Assuming the rocket was launched from ground level, about how high did it go?

b) Assuming the rocket was launched from ground level, about how high was the rocket 12 seconds after launch?

c) What is the rocket's acceleration at $t = 6$ seconds? At $t = 2$ seconds?

1091. The graph of a function f consists of a semicircle and two line segments as shown below. Let $g(x) = \int_1^x f(t)\, dt$.



a) Find $g(1)$.

b) Find $g(3)$.

c) Find $g(-1)$.

d) Find all the values of x on the open interval $(-3, 4)$ at which g has a relative maximum.

e) Write an equation for the line tangent to the graph of g at $x = -1$.

f) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.

g) Find the range of g.

1092. An automobile accelerates from rest at $1 + 3\sqrt{t}$ miles per hour per second for 9 seconds.

a) What is its velocity after 9 seconds?

b) How far does it travel in those 9 seconds?

1093. Find the function f with derivative $f'(x) = \sin x + \cos x$ whose graph passes through the point $(\pi, 3)$.

We have knowledge of the past but cannot control it; we may control the future but not have knowledge of it. —*Claude Shannon*

1094 (1989BC). Let f be a function such that $f''(x) = 6x + 8$.

a) Find $f(x)$ if the graph of f is tangent to the line $3x - y = 2$ at the point $(0, -2)$.

b) Find the average value of $f(x)$ on the closed interval $[-1, 1]$.

1095 (1999AB, Calculator). A particle moves along the y-axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$.

a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?

b) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at $t = 1.5$?

c) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.

d) Find the total distance traveled by the particle from $t = 0$ and $t = 2$.

1096 (1990BC). Let f and g be continuous functions with the following properties:

$$\begin{array}{rl} \text{i)} & g(x) = A - f(x) \text{ where } A \text{ is a constant} \\ \text{ii)} & \displaystyle\int_1^2 f(x)\,dx = \int_2^3 g(x)\,dx \\ \text{iii)} & \displaystyle\int_2^3 f(x)\,dx = -3A \end{array}$$

a) Find $\displaystyle\int_1^3 f(x)\,dx$ in terms of A.

b) Find the average value of $g(x)$ in terms of A over the interval $[1, 3]$.

c) Find the value of k if $\displaystyle\int_0^1 f(x+1)\,dx = kA$.

1097 (1994AB, Calculator). Let $F(x) = \displaystyle\int_0^x \sin(t^2)\,dt$ for $0 \leq x \leq 3$.

a) Use the trapezoidal rule with four equal subdivisions of the closed interval $[0, 1]$ to approximate $F(1)$.

b) On what interval is F increasing?

c) If the average rate of change of F on the closed interval $[1, 3]$ is k, find $\displaystyle\int_1^3 \sin(t^2)\,dt$ in terms of k.

1098 (1991BC). A particle moves on the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 12t^2 - 36t + 15$.

a) Find the position $x(t)$ of the particle at any time $t \geq 0$.

b) Find all values of t for which the particle is at rest.

c) Find the maximum velocity of the particle for $0 \leq t \leq 2$.

d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

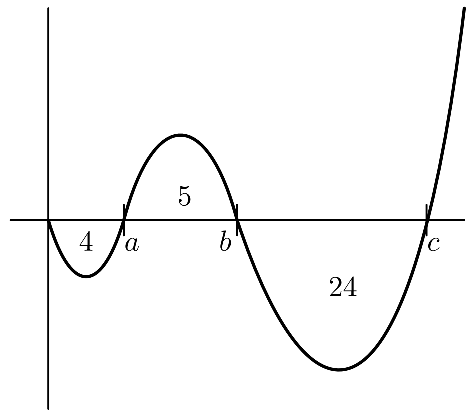
1099. A particle moves along the x-axis. Its initial position at $t = 0$ sec is $x(0) = 15$. The graph below shows the particle's velocity $v(t)$. The numbers are areas of the enclosed figures.

a) What is the particle's displacement between $t = 0$ and $t = c$?

b) What is the total distance traveled by the particle in the same time period?

c) Give the positions of the particle at times a, b, and c.

d) Approximately where does the particle achieve its greatest positive acceleration on the interval $[0, b]$? On $[0, c]$?



1100 (1987BC). Let f be a continuous function with domain $x > 0$ and let F be the function given by $F(x) = \int_1^x f(t)\,dt$ for $x > 0$. Suppose that $F(ab) = F(a) + F(b)$ for all $a > 0$ and $b > 0$ and that $F'(1) = 3$.

a) Find $f(1)$.

b) Prove that $aF'(ax) = F'(x)$ for every positive constant a.

c) Use the results from parts (a) and (b) to find $f(x)$. Justify your answer.

1101 (1999AB, Calculator). The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table below shows the rate as measured every 3 hours for a 24-hour period.

t (hours)	0	3	6	9	12	15	18	21	24
$R(t)$ (gal/hr)	9.6	10.4	10.8	11.2	11.4	11.3	10.7	10.2	9.6

a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate the value of $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

c) The rate of the water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}(768 + 23t - t^2)$. Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period. Indicate units of measure.

E. H. Moore was presenting a paper on a highly technical topic to a large gathering of faculty and graduate students from all parts of the country. When half way through he discovered what seemed to be an error (though probably no one else in the room observed it). He stopped and re-examined the doubtful step for several minutes and then, convinced of the error, he abruptly dismissed the meeting – to the astonishment of most of the audience. It was an evidence of intellectual courage as well as honesty and doubtless won for him the supreme admiration of every person in the group – an admiration which was in no ways diminished, but rather increased, when at a later meeting he announced that after all he had been able to prove the step to be correct. —*H. E. Slaught*

4.19 Multiple Choice Problems on Integrals

1102 (AP). For any real number b, $\displaystyle\int_0^b |2x|\,dx$ is

A) $-b|b|$ B) b^2 C) $-b^2$ D) $b|b|$ E) None of these

1103 (AP). Let f and g have continuous first and second derivatives everywhere. If $f(x) \le g(x)$ for all real x, which of the following must be true?

I) $f'(x) \le g'(x)$ for all real x
II) $f''(x) \le g''(x)$ for all real x
III) $\displaystyle\int_0^1 f(x)\,dx \le \int_0^1 g(x)\,dx$

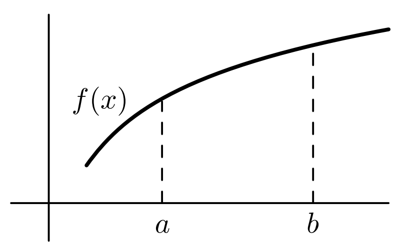
A) None B) I only C) III only D) I and II E) I, II, and III

1104 (AP). Let f be a continuous function on the closed interval $[0, 2]$. If $2 \le f(x) \le 4$, then the greatest possible value of $\displaystyle\int_0^2 f(x)\,dx$ is

A) 0 B) 2 C) 4 D) 8 E) 16

1105 (AP). If f is the continuous, strictly increasing function on the interval $[a, b]$ as shown below, which of the following must be true?

I) $\displaystyle\int_a^b f(x)\,dx < f(b)(b-a)$
II) $\displaystyle\int_a^b f(x)\,dx > f(a)(b-a)$
III) $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a)$ for some c in $[a, b]$.



A) I only B) II only C) III only D) I and II E) I, II, and III

1106 (AP). Which of the following definite integrals is *not* equal to zero?

A) $\displaystyle\int_{-\pi}^{\pi} \sin^3 x\,dx$ B) $\displaystyle\int_{-\pi}^{\pi} x^2 \sin x\,dx$ C) $\displaystyle\int_0^{\pi} \cos x\,dx$

D) $\displaystyle\int_{-\pi}^{\pi} \cos^3 x\,dx$ E) $\displaystyle\int_{-\pi}^{\pi} \cos^2 x\,dx$

Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. —*G. H. Hardy*

1107. $\displaystyle\int_{\pi/6}^{\pi/2} \cot x \; dx =$

A) $\ln \frac{1}{2}$ B) $\ln 2$ C) $\frac{1}{2}$ D) $\ln(\sqrt{3}-1)$ E) None of these

1108. $\displaystyle\int_{-2}^{3} |x+1| \; dx =$

A) $\frac{5}{2}$ B) $\frac{17}{2}$ C) $\frac{9}{2}$ D) $\frac{11}{2}$ E) $\frac{13}{2}$

1109. $\displaystyle\int_{1}^{2} (3x-2)^3 \; dx =$

A) $\frac{16}{3}$ B) $\frac{63}{4}$ C) $\frac{13}{3}$ D) $\frac{85}{4}$ E) None of these

1110. $\displaystyle\int_{\pi/4}^{\pi/2} \sin^3\theta \cos\theta \; d\theta =$

A) $\frac{3}{16}$ B) $\frac{1}{8}$ C) $-\frac{1}{8}$ D) $-\frac{3}{16}$ E) $\frac{3}{4}$

1111. $\displaystyle\int_{0}^{1} \frac{e^x}{(3-e^x)^2} \; dx =$

A) $3\ln(e-3)$ B) 1 C) $\dfrac{1}{3-e}$ D) $\dfrac{e-1}{2(3-e)}$ E) $\dfrac{e-2}{3-e}$

1112. $\displaystyle\int_{-1}^{0} e^{-x} \; dx =$

A) $1-e$ B) $\dfrac{1-e}{e}$ C) $e-1$ D) $1-\dfrac{1}{e}$ E) $e+1$

1113. $\displaystyle\int_{0}^{1} \frac{x}{x^2+1} \; dx =$

A) $\frac{\pi}{4}$ B) $\ln\sqrt{2}$ C) $\frac{1}{2}(\ln 2 - 1)$ D) $\frac{3}{2}$ E) $\ln 2$

Anyone who cannot cope with mathematics is not fully human. At best he is a tolerable subhuman who has learned to wear shoes, bathe, and not make messes in the house. —*Robert A. Heinlein*

1114. The acceleration of a particle moving along a straight line is given by $a = 6t$. If, when $t = 0$ its velocity $v = 1$ and its distance $s = 3$, then at any time t the position function is given by

A) $s = t^3 + 3t + 1$

B) $s = t^3 + 3$

C) $s = t^3 + t + 3$

D) $s = \frac{1}{3}t^3 + t + 3$

E) $s = \frac{1}{3}t^3 + \frac{1}{2}t^2 + 3$

1115. If the displacement of a particle on a line is given by $s = 3 + (t-2)^4$, then the number of times the particle changes direction is

A) 0 B) 1 C) 2 D) 3 E) None of these

1116. $\displaystyle\int_0^{\pi/2} \cos^2 x \sin x \; dx =$

A) -1 B) $-\frac{1}{3}$ C) 0 D) $\frac{1}{3}$ E) 1

1117. $\displaystyle\int_0^1 (3x^2 - 2x + 3) \; dx =$

A) 0 B) 5 C) 3 D) 8 E) None of these

1118. $\displaystyle\int_1^e \left(x - \frac{1}{2x}\right) \; dx =$

A) $\frac{1}{2}e^2$ B) $\frac{1}{2}e^2 + 1$ C) $\frac{1}{2}(e^2 + 1)$ D) $\frac{1}{2}(e^2 - 1)$ E) None of these

1119. $\displaystyle\int_0^1 (2 - 3x)^5 \; dx =$

A) $-\frac{1}{2}$ B) $\frac{1}{6}$ C) $\frac{1}{2}$ D) $-\frac{1}{18}$ E) None of these

Work is the greatest thing in the world, so we should always save some of it for tomorrow. —*Don Herald*

A.P. Calculus Test Four
Section One
Multiple-Choice
No Calculators

Time—35 minutes
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where C is the number of correct responses and I is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1. $\displaystyle\int \sin 3\theta \; d\theta =$

A) $3\cos 3\theta + C$

B) $-3\cos 3\theta + C$

C) $-\cos 3\theta + C$

D) $\frac{1}{3}\cos 3\theta + C$

E) $-\frac{1}{3}\cos 3\theta + C$

---

2. $\displaystyle\int 3^{x^2} x \; dx =$

A) $\dfrac{3^{x^2+1}}{x^2+1} + C$

B) $\dfrac{3^{x^2}}{\ln 9} + C$

C) $3^{x^2} \ln 3 + C$

D) $3^{x^3/3} + C$

E) None of these

---

3. Let $f(x)$ be defined as below. Evaluate $\displaystyle\int_0^6 f(x) \; dx$.

$$f(x) = \begin{cases} x & 0 < x \le 2 \\ 1 & 2 < x \le 4 \\ \frac{1}{2}x & 4 < x \le 6 \end{cases}$$

A) 5

B) 6

C) 7

D) 8

E) 9

4. $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx =$

A) $\frac{\pi}{4}$

B) $\ln\sqrt{2}$

C) $\frac{1}{2}(\ln 2 - 1)$

D) $\frac{3}{2}$

E) $\ln 2$

---

5. The average value of $g(x) = (x-3)^2$ in the interval $[1,3]$ is

A) 2

B) $\frac{2}{3}$

C) $\frac{4}{3}$

D) $\frac{8}{3}$

E) None of these

---

6. $\displaystyle\int_0^5 \frac{dx}{\sqrt{3x+1}} =$

A) $\frac{1}{2}$

B) $\frac{2}{3}$

C) 1

D) 2

E) 6

7. There is a point between $P(1,0)$ and $Q(e,1)$ on the graph of $y = \ln x$ such that the tangent to the graph at that point is parallel to the line through points P and Q. The x-coordinate of this point is

A) $e - 1$

B) e

C) -1

D) $\dfrac{1}{e-1}$

E) $\dfrac{1}{e+1}$

---

8. Which of the following statements are true?

I. If the graph of a function is always concave up, then the left-hand Riemann sums with the same subdivisions over the same interval are always less than the right-hand Riemann sum.

II. If the function f is continuous on the interval $[a,b]$ and $\int_a^b f(x)\,dx = 0$, then f must have at least one zero between a and b.

III. If $f'(x) > 0$ for all x in an interval, then the function f is concave up in that interval.

A) I only

B) II only

C) III only

D) II and III only

E) None are true.

---

9. If $f(x) = \displaystyle\int_2^{2x} \frac{1}{\sqrt{t^3+1}}\,dt$, then $f'(1) =$

A) 0

B) $\frac{1}{3}$

C) $\frac{2}{3}$

D) $\sqrt{2}$

E) undefined

10. If $\int_a^b f(x)\,dx = 3$ and $\int_a^b g(x)\,dx = -2$, then which of the following must be true?

$$\begin{array}{rl} \text{I.} & f(x) > g(x) \text{ for all } a \le x \le b \\ \text{II.} & \int_a^b [f(x) + g(x)]\,dx = 1 \\ \text{III.} & \int_a^b [f(x)g(x)]\,dx = -6 \end{array}$$

A) I only

B) II only

C) III only

D) II and III only

E) I, II, and III

---

11. The graph of f is shown below. Approximate $\int_{-3}^{3} f(x)\,dx$ using the trapezoid rule with 3 equal subdivisions.

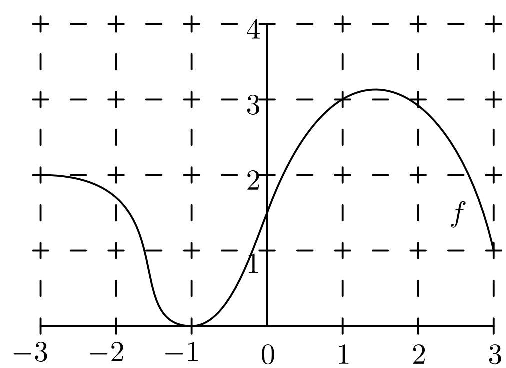
A) $\frac{9}{4}$

B) $\frac{9}{2}$

C) 9

D) 18

E) 36



---

12. If $\int_0^k \frac{\sec^2 x}{1 + \tan x}\,dx = \ln 2$, then the value of k is

A) $\pi/6$.

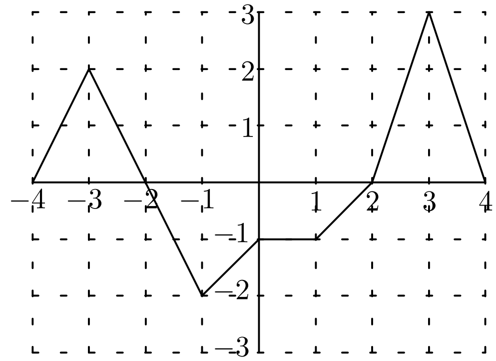
B) $\pi/4$.

C) $\pi/3$.

D) $\pi/2$.

E) π.

13. The graph of the function f on the interval $[-4, 4]$ is shown below. $\int_{-4}^{4} |f(x)|\, dx =$



A) 1

B) 2

C) 5

D) 8

E) 9

---

14. The acceleration of a particle moving along the x-axis at time $t > 0$ is given by $a(t) = \dfrac{1}{t^2}$. When $t = 1$ second, the particle is at $x = 2$ and has velocity -1 unit per second. If $x(t)$ is the particle's position, then the position when $t = e$ seconds is

A) $x = -2$.

B) $x = -1$.

C) $x = 0$.

D) $x = 1$.

E) $x = 2$.

---

15. The area enclosed by the two curves $y = x^2 - 4$ and $y = x - 4$ is given by

A) $\int_0^1 (x - x^2)\, dx$

B) $\int_0^1 (x^2 - x)\, dx$

C) $\int_0^2 (x - x^2)\, dx$

D) $\int_0^2 (x^2 - x)\, dx$

E) $\int_0^4 (x^2 - x)\, dx$

A.P. Calculus Test Four
Section Two
Free-Response
Calculators Allowed

Time—45 minutes

Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. The temperature on New Year's Day in Buffalo, New York, is given by

$$T(h) = -A - B\cos\left(\frac{\pi h}{12}\right),$$

where T is the temperature in degrees Fahrenheit and h is the number of hours from midnight ($0 \le h \le 24$).

a) The initial temperature at midnight was $-15°$ F, and at Noon of New Year's Day it was $5°$ F. Find A and B.

b) Find the average temperature for the first 10 hours.

c) Use the trapezoid rule with 4 equal subdivisions to estimate $\int_6^{10} T(h)\, dh$. Using correct units, explain the meaning of your answer.

d) Find an expression for the rate that the temperature is changing with respect to h.

---

2. Let f be a differentiable function, defined for all real numbers x, with the following properties.

1) $f'(x) = ax^2 + bx$, where a and b are real numbers
2) $f'(1) = 6$ and $f''(1) = 18$
3) $\int_1^2 f(x)\, dx = 18$

Find $f(x)$. Show your work.

---

3. A particle moves along the x-axis so that its velocity at any time $t \ge 0$ is given by $v(t) = 1 - \sin(2\pi t)$.

a) Find the acceleration $a(t)$ of the particle at any time t.

b) Find all values of t, $0 \le t \le 2$, for which the particle is at rest.

c) Find the position $x(t)$ of the particle at any time t if $x(0) = 0$.

CHAPTER 5

APPLICATIONS of INTEGRALS

5.1 Volumes of Solids with Defined Cross-Sections

1120. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 9$ and the cross sections perpendicular to the x-axis are squares.

1121. Find the volume of the solid whose base is bounded by the ellipse $x^2 + 4y^2 = 4$ and the cross sections perpendicular to the x-axis are squares.

1122. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 1$ and the cross sections perpendicular to the x-axis are equilateral triangles.

1123. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 4$ and the cross sections perpendicular to the x-axis are semicircles.

1124. Find the volume of the solid whose base is bounded by the circle $x^2 + y^2 = 16$ and the cross sections perpendicular to the x-axis are isosceles right triangles having the hypotenuse in the plane of the base.

1125. Let R be the region bounded by $y = e^x$, $y = 2$, and $x = 0$. Find the volume of the solid whose base is bounded by the region R and the cross sections perpendicular to the x-axis are semicircles.

1126. Let R be the region bounded by $y = e^x$, $y = 2$, and $x = 0$. Find the volume of the solid whose base is bounded by the region R and the cross sections perpendicular to the x-axis are quartercircles.

1127. Let R be the region bounded by $y = x^2$ and $y = x$. Find the volume of the solid whose base is bounded by the region R and the cross sections perpendicular to the x-axis are semicircles.

1128. Let R be the region bounded by $y = \frac{1}{16}x^2$ and $y = 2$. Find the volume of the solid whose base is bounded by the region R and the cross sections perpendicular to the x-axis are rectangles whose height is twice that of the side in the plane of the base.

1129. Find the volume of the solid whose base is bounded by the curve $y = 2\sqrt{\sin x}$, the lines $x = 0$, $x = \pi$, and $y = 0$, and the cross sections perpendicular to the x-axis are a) equilateral triangles; b) squares.

1130. Find the volume of the solid whose base is bounded by the curve $y = 2x^3$, the line $x = 2$ and the line $y = 0$, and the cross sections perpendicular to the x-axis are equilateral triangles.

1131. Find the volume of the solid whose base is bounded by the curve $y = 2x^3$, the line $x = 2$ and the line $y = 0$, and the cross sections perpendicular to the y-axis are equilateral triangles.

In most sciences one generation tears down what another has built, and what one has established another undoes. In mathematics alone each generation adds a new storey to the old structure. —*Hermann Hankel*

5.2 Turn Up the Volume!

Sketch the region R bounded by the given curves and lines. Then find the volume of the solid generated by revolving R around the given axis.

1132. $y = -2/x,\ y = 1,\ y = 2,\ x = 0;$ axis: $x = 0$

1133. $y = x^2,\ y = 2 - x^2;$ axis: y-axis

1134. $y = \cos x,\ y = \sin x,\ x = 0,\ x = \pi/4;$ axis: x-axis

1135. $y = x^2,\ y = 0,\ x = 2;$ axis: y-axis

1136. $y = 1/x^2,\ x = e,\ x = e^3,\ y = 0;$ axis: x-axis

1137. $y = 1/x^2,\ x = e,\ x = e^3,\ y = 0;$ axis: y-axis

1138. $y = 3 - x^2,\ y = -1;$ axis: $y = -1$

1139. $x = 1 - y^2,\ x = -3;$ axis: $x = -3$

1140. $y = 16x - 4x^2,\ y = 0;$ axis: $y = -20$

1141. $y = (x+3)^3,\ y = 0,\ x = 2;$ axis: $y = -1$

Set up the integrals that represent the volume of the solid described in the following problems. Then use your calculator to evaluate the integrals.

1142. The region R is bounded by the curve $y = -\frac{1}{2}x^3$ and the lines $y = 4$ and $x = 1$. Find the volume of the solid generated by revolving R about the axis

a) $x = 2$ b) $y = 5$ c) $x = -3$ d) $y = -\frac{3}{2}$

1143. The region R is bounded by the curve $y = \sin x \cos x$ and the x-axis from $x = 0$ to $x = \pi/2$. Find the volume of the solid generated by revolving R about the x-axis.

1144. The region R is bounded by the curve $y = \ln x$ and the lines $y = 0$ and $x = e^3$. Find the volume of the solid generated by revolving R about the y-axis.

1145. The region R is bounded by the curve $y = e^x$ and the lines $y = 2$ and $x = -1$. Find the volume of the solid generated by revolving R about the line $y = e$.

1146. The region R is bounded by the curve $16y^2 + 9x^2 = 144$ and the line $4y = 3x + 12$ in Quadrant II. Find the volume of the solid generated by revolving R about the x-axis.

1147. The arch $y = \sin x,\ 0 \le x \le \pi$, is revolved about the line $y = c$, for $0 \le c \le 1$, to generate a solid. Find the value of c that minimizes the volume of the solid. What is the minimum volume? What value of c in $[0, 1]$ maximizes the volume of the solid?

In the index to the six hundred odd pages of Arnold Toynbee's *A Study of History*, abridged version, the names of Copernicus, Galileo, Descartes and Newton do not occur yet their cosmic quest destroyed the medieval vision of an immutable social order in a walled-in universe and transformed the European landscape, society, culture, habits and general outlook, as thoroughly as if a new species had arisen on this planet. —*Arthur Koestler*

5.3 Volume and Arc Length

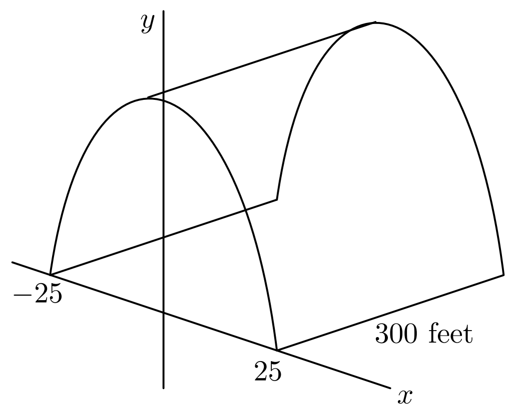
1148. Find the length of the curve $x^2 + y^2 = 1$ using two different approaches. One of the techniques must involve an integral.

1149. Set up, *but do not evaluate,* an integral that would represent the length of the ellipse $9x^2 + 4y^2 = 36$ in Quadrant I.

1150. Set up, *but do not evaluate,* an integral that would represent the length of the hyperbola $4x^2 - 25y^2 = 100$ in Quadrant I from $x = 0$ to $x = 7$.

1151. Set up, *but do not evaluate,* an integral that would represent the length of the curve $y = \int_0^x \tan t \, dt$ from $x = 0$ to $x = \pi/6$.

1152. Your engineering firm is bidding for the contract to construct the tunnel shown in the figure. The tunnel is 300 ft long and 50 ft across at the base. It is shaped like one arch of the curve $y = 25\cos\frac{\pi x}{50}$. Upon completion, the tunnel's inside surface (excluding the roadway) will be coated with a waterproof sealant that costs \$1.75 per square foot to apply. How much will it cost to apply the sealant?



Find the exact length of the given curve.

1153. $y = x^{3/2}$ from $x = 0$ to $x = 3$

1154. $y = \frac{2}{3}(x+3)^{3/2}$ from $x = 1$ to $x = 6$

1155. $y = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$

1156. $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 5$ from $x = 1$ to $x = 8$

1157. $y = \int_{-2}^{x} \sqrt{3t^4 - 1} \, dt$ from $x = -2$ to $x = 1$

Sketch the region R bounded by the given curves, lines, and the x-axis. Then find the volume of the solid generated by revolving R around the x-axis.

1158. $f(x) = \sqrt{x-2},\ x = 3,\ x = 4$

1159. $f(x) = x^3 + 8,\ x = 0$

1160. $f(x) = \sin 2x,\ x = 0,\ x = \pi$

1161. $f(x) = \frac{1}{2}e^{x/8},\ x = \ln 16,\ x = \ln 81$

1162. $f(x) = 3/x,\ x = e,\ x = 3$

1163. $f(x) = 2\cos 3x,\ x = \pi/6,\ x = \pi/3$

1164. $f(x) = x^2,\ g(x) = x$

1165. $f(x) = 1/\sqrt{x},\ x = 1,\ x = e$

5.4 Differential Equations, Part One

Find the function with the given derivative whose graph passes through the point P.

1166. $f'(x) = 2x - 1, \;\; P(0,0)$

1167. $g'(x) = \dfrac{1}{x} + 2x, \;\; P(1,-1)$

1168. $f'(x) = e^{2x}, \;\; P(0, \frac{3}{2})$

1169. $r'(t) = \sec t \tan t - 1, \;\; P(0,0)$

1170. $s'(t) = 9.8t + 5, \;\; P(0,10)$

1171. $s'(t) = 32t - 2, \;\; P(0.5,4)$

Given the acceleration, initial velocity, and initial position of a particle, find the particle's position at any time t.

1172. $a(t) = e^t, \;\; v(0) = 20, \;\; s(0) = 5$

1173. $a(t) = -4\sin(2t), \;\; v(0) = 2, \;\; s(0) = -3$

Find the general solution to the given differential equation.

1174. $\dfrac{dy}{dx} = 2x + 7$

1175. $\dfrac{dy}{dx} = 4x^3 + 2x - 1$

1176. $\dfrac{dr}{dt} = 4t^3 r$

1177. $\dfrac{du}{dv} = 2u^4 v$

1178. $\dfrac{dy}{dx} = 2\sqrt{x}$

1179. $\dfrac{dy}{dx} = 2(3x+5)^3$

Find the particular solution to the given differential equation.

1180. $\dfrac{ds}{dt} = \cos t + \sin t, \;\; s(\pi) = 1$

1181. $\dfrac{dr}{d\theta} = -\pi \sin(\pi\theta), \;\; r(0) = 0$

1182. $\dfrac{dv}{dt} = \dfrac{3}{t\sqrt{t^2-1}}, \;\; v(2) = 0, \;\; t > 1$

1183. $\dfrac{dv}{dt} = \dfrac{8}{1+t^2} + \sec^2 t, \;\; v(0) = 1$

1184. $\dfrac{d^2y}{dx^2} = 2 - 6x, \;\; y'(0) = 4, \;\; y(0) = 1$

1185. $\dfrac{d^2y}{dx^2} = \dfrac{2}{x^3}, \;\; y'(1) = 1, \;\; y(1) = 1$

1186. *A Curious Property of Definite Integrals*

a) Let R_1 be the region bounded by $f(x) = \frac{1}{x}$, $x = 1$, $x = 3$, and the x-axis. Draw the region R_1 and find the area of of R_1 using an integral.

b) Let R_2 be the region bounded by $f(x) = \frac{1}{x-2}$, $x = 3$, $x = 5$, and the x-axis. Draw the region R_2 and find the area of of R_2 using an integral.

c) What do you notice about your answers in parts (a) and (b)?

d) Complete the following conjecture, where a, b, and c are real numbers: *If $f(x)$ is a continuous function on $[a,b]$, then* $\displaystyle\int_a^b f(x)\,dx = \int_{a+c}^{b+c} \_\_\_\_\_\_ \,dx.$

5.5 The Logistic Curve

1187. The graph of a function of the form $P(t) = \dfrac{M}{1 + Ce^{-rMt}}$, where M, r, and C are constants, is called a *logistic* curve. Graph the function $y(x) = \dfrac{8}{1 + 10e^{-0.9x}}$ in the window $-1 \leq x \leq 10$, $-1 \leq y \leq 9$. What value does y approach as $x \to \infty$? What appears to be the y-value of the point where dy/dt is changing the fastest?

1188. The solution to the differential equation $\dfrac{dP}{dt} = r(M - P)P$ is a logistic curve, where C is determined by the initial condition. Can the values found in the previous problem be found without solving the differential equation? In other words, in the equation $dP/dt = 0.001(100 - P)P$, what does P approach as $x \to \infty$? What appears to be the P-value of the point where dP/dt is changing the fastest?

1189. A 2000 gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is $dP/dt = 0.0015(150 - P)P$, where t is in weeks. Find a formula for the guppy population in terms of t; then, determine how long it will take for the guppy population to be 100.

1190. A certain wild animal preserve can support no more than 250 gorillas. In 1970, 28 gorillas were known to be in the preserve. Assume that the rate of growth of population is $dP/dt = 0.0004(250 - P)P$, where t is in years. Find a formula for the gorilla population in terms of t; then, determine how long it will take for the gorilla population to reach the carrying capacity of the preserve. What is the gorilla population when the rate of change of the population is maximized?

1191. Solve the differential equation $dP/dt = kP^2$ for constant k, with initial condition $P(0) = P_0$. Prove that the graph of the solution has a vertical asymptote at a positive value of t. What is that value of t? (This value is called the *catastrophic solution*.)

1192. Given a differential equation of the form $ay'' + by' + y = 0$, find constants a and b so that both $y = e^x$ and $y = e^{2x}$ are solutions.

1193 (AP). At each point (x, y) on a certain curve, the slope of the curve is $3x^2y$. If the curve contains the point $(0, 8)$, then its equation is

A) $y = 8e^{x^3}$

B) $y = x^3 + 8$

C) $y = e^{x^3} + 7$

D) $y = \ln(x + 1) + 8$

E) $y^2 = x^2 + 8$

The simplest schoolboy is now familiar with facts for which Archimedes would have sacrificied his life. —*Earnest Renan*

5.6 Differential Equations, Part Two

1194. You are driving along the highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant decceleration is required to stop your car in 242 ft?

1195. The rate of change in the number of bacteria in a culture is proportional to the number present. AP Biology students at Rockdale discovered that there were 3000 bacteria initially, and 90,000 bacteria after two hours.

a) In terms of t only, find the number of bacteria in the culture at any time t.

b) How many bacteria were there after four hours?

c) How many hours have elapsed when the students observed 60,000 bacteria in the culture?

1196. The rate of increase of the population of Springfield is proportional to the population at any given time. If the population in 1950 was 50,000 and in 1980 it was 75,000, what is the expected population in the year 2010? When will Springfield's population reach 1,000,000 people? Justify your answer.

1197. Corbin's hobby is to buy antique cars, repair them, and then sell them at a good profit. Research shows that the rate of change in the value of Corbin's cars is directly proportional to the value of the car at any given time. If Corbin bought a 1945 Jaguar from his aunt for \$49,000 in 2002, and if the Jaguar's market value in 2008 is \$63,000, what is the expected value of the Jaguar in the year 2014? How long will Corbin have to wait for the Jaguar's market value to be \$100,000?

1198. Oil is being pumped continuously from an Arabian oil well at a rate proportional to the amount of oil left in the well; that is, $dy/dt = ky$ where y is the number of gallons of oil left in the well at any time t (in years). Initially there are 1,000,000 gallons of oil in the well, and 6 years later there are 500,000 remaining. Assume that it is no longer profitable to pump oil when there are fewer than 50,000 gallons remaining.

a) Write an equation for y in terms of t.

b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining in the well?

c) How long will it be profitable to pump oil?

1199. When stated in the form of a differential equation, *Newton's Law of Cooling* becomes $dT/dt = -k(T - T_a)$, where k is a positive constant and T_a is the ambient temperature.

a) Find the general solution for T, satisfying the initial condition $T(0) = T_0$.

b) What is the limiting temperature as t? Explain the difference between what happens when $T_0 \leq T_a$, and when $T_0 \geq T_a$.

c) A 15-pound roast, initially at 60°F, is put into a 350°F oven at 2 PM. After two hours, the temperature of the roast is 120°F. When will the roast be at a temperature of 150°F?

5.7 Slope Fields and Euler's Method

1200. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{2x}{y}$.

a) Sketch a slope field for the equation at the points $(-1,1)$, $(-1,2)$, $(-1,3)$, $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, and $(1,3)$.

b) Use the slope field program on your calculator to generate a slope field.

c) Solve the equation and sketch the solution curve through the points $(4,6)$ and $(-4,6)$.

1201. Consider the ellipse $4x^2 + 9y^2 = 36$.

a) Find dy/dx.

b) Graph the slope field for the differential equation found in part (a) using your calculator.

c) Graph the particular solution passing through $(3,0)$.

1202. Let $\dfrac{dy}{dx} = -\dfrac{2x}{y}$ be a differential equation that contains the point $(0, 2\sqrt{2})$.

a) Approximate 6 points in the particular solution to the above equation using Euler's Method. Use 0.2 as the step size. Do not use your calculator program.

b) Repeat part (a) with a step size of 0.1, but this time, use your calculator program.

c) Solve the equation analytically. Compare the actual y-values with those obtained using Euler's Method. What conclusion could you draw?

1203. Let $\dfrac{dy}{dx} = \dfrac{3}{x}$ be subject to the initial condition that $y(1) = 0$.

a) Use Euler's Method with step size 0.25 to approximate $y(2)$. Do not use your calculator.

b) Solve the equation and calculate the exact value of $y(2)$.

c) Graph the slope field for the equation and use it to determine if you answer in part (a) is greater than or less than the value obtained in part (b).

1204. The *normal probability density function* is very important in statistics and is defined by $G(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^x e^{-t^2/2}\, dt$.

a) Find $G'(x)$.

b) Use the slope field program to help you sketch the slope field for $G'(x)$. Use a window of $-4 \le x \le 4$ by $-1 \le y \le 1$.

c) Sketch the solution curve through the point $(0, \frac{1}{2})$.

God does not play dice with the universe. —*Albert Einstein*

5.8 Differential Equations, Part Three

1205. The growth rate of an evergreen shrub during its first 6 years is approximated by $dh/dt = 1.5t + 6$, where t is the time in years and h is the height in centimeters. The seedlings are 12 cm tall when planted ($t = 0$). Find the height after t years; then, determine the height of the shrubs after 6 years.

1206. The rate of growth of a population of bacteria is proportional to the square root of t, where P is the population and t is the time in days for $0 \leq t \leq 10$. The initial size of the population is 500. After 1 day, the population has grown to 600. Estimate the population after 7 days.

1207. Suppose that rabbits are introduced onto an island where they have no natural enemies. Because of natural conditions, the island can support a maximum of 1000 rabbits. Let $P(t)$ denote the number of rabbits at time t (measured in months), and suppose that the population varies in size (due to births and deaths) at a rate proportional to both $P(t)$ and $1000 - P(t)$. That is, suppose that $P(t)$ satisfies the differential equation $dP/dt = kP(1000 - P)$, where k is a positive constant.

a) Find the value of P when the rate of change of the rabbit population is maximized.

b) When is the rate of change of the rabbit population a minimum? Discuss your answers.

c) Assuming 50 rabbits were placed on the island, sketch the graph that would show how t and $P(t)$ are related.

1208. Show that $y = x^3 + x + 2 + \displaystyle\int_0^x \sin(t^2)\, dt$ is a solution to the differential equation $y'' = 6x + 2x\cos(x^2)$ with initial conditions $y'(0) = 1$ and $y(0) = 2$.

1209. Under some conditions, the result of the movement of a dissolved substance across a cell's membrane is described by the differential equation $\dfrac{dy}{dt} = k\dfrac{A}{V}(c - y)$, where y is the concentration of the substance in the cell, A is the surface area of the membrane, V is the cell's volume, c the concentration of the substance outside the cell, and k is a constant. Solve the equation with initial condition $y(0) = y_0$; then, determine $\lim\limits_{t\to\infty} y(t)$. (This is called the *steady state* concentration.)

God not only plays dice with the universe, He sometimes tosses them where we cannot see them. —*Stephen Hawking*

5.9 Sample A.P. Problems on Applications of Integrals

For the following four problems, use Euler's Method with the given step size h to estimate the the value of the solution at the given point x^*.

1210. $y' = 2xe^{x^2}$, $y(0) = 2$, $h = 0.1$, $x^* = 1$

1211. $y' = y + e^x - 2$, $y(0) = 2$, $h = 0.5$, $x^* = 2$

1212. $y' = y^2/\sqrt{x}$, $y(1) = -1$, $h = 0.5$, $x^* = 5$

1213. $y' = y - e^{2x}$, $y(0) = 1$, $h = \frac{1}{3}$, $x^* = 2$

1214. Let R represent the area in Quadrant IV bounded by $f(x) = x^3 - 4x$ and $g(x) = 0$.

a) Find the area of R.

b) Find the volume of the solid generated by revolving R around the x-axis.

c) Find the average value of $f(x)$ over the interval $[-3, -2]$.

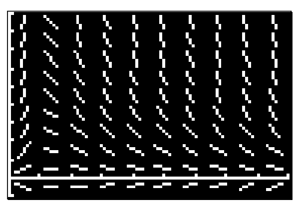
1215. Match the differential equation with its slope field.

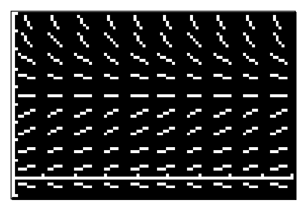
a) $\dfrac{dy}{dx} = 0.065y$

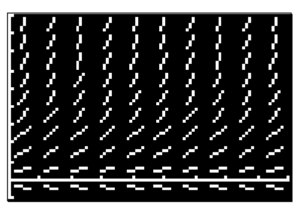
b) $\dfrac{dy}{dx} = 0.06y\left(1 - \dfrac{y}{100}\right)$

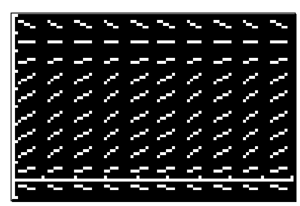
c) $\dfrac{dy}{dx} = \dfrac{y}{x} - y$

d) $\dfrac{dy}{dx} = 0.06y\left(1 - \dfrac{y}{150}\right)$


I)


III)


II)


IV)

1216 (1996AB). Let R be the region in the first quadrant under the graph of $y = \dfrac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

a) Find the area of R.

b) If the line $x = k$ divides the region R into two regions of equal area, what is the value of k?

c) Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are squares.

1217. Use your calculator to find the length of the curve $f(x) = \dfrac{x-1}{4x^2+1}$ on the interval $[-\frac{1}{2}, 1]$.

Small minds discuss persons. Average minds discuss events. Great minds discuss ideas. Really great minds discuss mathematics. —*Nicolai Lobachevsky*

1218 (1995AB, Calculator). The region R_1 is bounded above by $g(x) = 2^x$ and below by $f(x) = x^2$, while the region R_2 is bounded above by $f(x) = x^2$ and bounded below by $g(x) = 2^x$.

a) Find the x- and y-coordinates of the three points of intersection of the graphs of f and g.

b) Without using absolute value, set up an expression involving one or more integrals that gives the total area enclosed by the graphs of f and g. Do not evaluate.

c) Without using absolute value, set up an expression involving one or more integrals that gives the volume of the solid generated by revolving the region R_1 about the line $y = 5$. Do not evaluate.

1219 (1999BC, Calculator). Let f be the function whose graph goes through the point $(3, 6)$ and whose derivative is given by $f'(x) = \dfrac{1 + e^x}{x^2}$.

a) Write an equation of the line tangent to the graph of f at $x = 3$ and use it to approximate $f(3.1)$.

b) Use Euler's Method, starting at $x = 3$ with a step size of 0.05, to approximate $f(3.1)$. Use f'' to explain why this approximation is less than $f(3.1)$.

c) Use $\displaystyle\int_3^{3.1} f'(x)\,dx$ to evaluate $f(3.1)$.

1220. Find the particular solution to the differential equation $\dfrac{dy}{dx} = \dfrac{4\sqrt{y}\ln x}{x}$ with initial value $y(e) = 1$.

1221 (1997BC, Calculator). Let R be the region enclosed by the graphs of $y = \ln(x^2 + 1)$ and $y = \cos x$.

a) Find the area of R.

b) Write an expression involving one or more integrals that gives the length of the boundary of the region R. Do not evaluate.

c) The base of a solid is the region R. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

1222 (1993AB). Let $P(t)$ represent the number of wolves in a population at time t years, when $t \geq 0$. The population P is increasing at a rate directly proportional to $800 - P$, where the constant of proportionality is k.

a) If $P(0) = 500$, find $P(t)$ in terms of t and k.

b) If $P(2) = 700$, find k.

c) Find $\lim\limits_{t\to\infty} P(t)$.

From the intrinsic evidence of His creation, the Great Architect of the universe now begins to appear as a pure mathematician. —*Sir James Hopwood Jeans*

1223 (1993AB). Consider the curve $y^2 = 4+x$ and chord $\overline{AB}$ joining points $A(-4, 0)$ and $B(0, 2)$ on the curve.

a) Find the x- and y-coordinates of the point on the curve where the tangent line is parallel to chord $\overline{AB}$.

b) Find the area of the region R enclosed by the curve and chord $\overline{AB}$.

c) Find the volume of the solid generated when the region R defined in part (b) is revolved about the x-axis.

1224 (1988AB). Let R be the region in the first quadrant enclosed by the hyperbola $x^2 - y^2 = 9$, the x-axis, and the line $x = 5$.

a) Find the volume of the solid generated by revolving region R about the x-axis.

b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region R about the line $x = -1$.

1225 (1991BC, Calculator). Let $F(x) = \displaystyle\int_1^{2x} \sqrt{t^2 + t}\, dt$.

a) Find $F'(x)$.

b) Find the domain of F.

c) Find $\displaystyle\lim_{x\to 1/2} F(x)$.

d) Find the length of the curve $y = F(x)$ for $1 \le x \le 2$.

1226 (1989AB). Let R be the region in the first quadrant enclosed by the graph of $y = \sqrt{6x+4}$, the line $y = 2x$, and the y-axis.

a) Find the area of R.

b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region R about the x-axis.

c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving region R about the y-axis.

1227 (1990BC). Let R be the region in the xy-plane between the graphs of $y = e^x$ and $y = e^{-x}$ from $x = 0$ to $x = 2$.

a) Find the volume of the solid generated when R is revolved about the x-axis.

b) Find the volume of the solid generated when R is revolved about the y-axis.

We've all heard that a million monkeys banging on a million typewriters will eventually reproduce the entire works of Shakespeare. Now, thanks to the internet, we know this is not true. —*Robert Silensky*

5.10 Multiple Choice Problems on Application of Integrals

1228. $\displaystyle\int_1^e \frac{2}{1+3x}\,dx =$

A) $\dfrac{2}{3}\ln\left(\dfrac{1+3e}{4}\right)$

B) $\dfrac{-1}{3(1+3e)^2}+48$

C) $2\ln\left(\dfrac{1+3e}{4}\right)$

D) $\dfrac{3}{(1+3e)^2}-\dfrac{3}{16}$

E) None of these

1229 (AP)**.** Which of the following integrals gives the length of the graph of $y=\sqrt{x}$ on the interval $[a,b]$?

A) $\displaystyle\int_a^b \sqrt{x^2+x}\,dx$

B) $\displaystyle\int_a^b \sqrt{x+\sqrt{x}}\,dx$

C) $\displaystyle\int_a^b \sqrt{x+\frac{1}{2\sqrt{x}}}\,dx$

D) $\displaystyle\int_a^b \sqrt{1+\frac{1}{2\sqrt{x}}}\,dx$

E) $\displaystyle\int_a^b \sqrt{1+\frac{1}{4x}}\,dx$

1230. $\displaystyle\int_{\pi/4}^{\pi} \sin(2\theta)\,d\theta =$

A) -2 B) $\frac{1}{2}$ C) $-\frac{1}{2}$ D) 2 E) $\frac{3}{2}\sqrt{2}$

1231. The average value of $\sqrt{x}$ over the interval $[0,2]$ is

A) $\frac{1}{3}\sqrt{2}$ B) $\frac{1}{2}\sqrt{2}$ C) $\frac{2}{3}\sqrt{2}$ D) 1 E) $\frac{4}{3}\sqrt{2}$

1232. Estimate the area bounded by $y=x^2$, the x-axis, the line $x=1$ and the line $x=2$ by using a left-hand Riemann sum with 3 subintervals of equal length.

A) $\frac{50}{27}$ B) $\frac{251}{108}$ C) $\frac{7}{3}$ D) $\frac{127}{54}$ E) $\frac{77}{27}$

1233. If $dy/dx=\cos(2x)$, then $y=$

A) $-\frac{1}{2}\cos(2x)+C$

B) $-\frac{1}{2}\cos^2 x+C$

C) $\frac{1}{2}\sin(2x)+C$

D) $\frac{1}{2}\sin^2(2x)+C$

E) $-\frac{1}{2}\sin(2x)+C$

One of the penalties for refusing to participate in politics is that you end up being governed by your inferiors. —*Plato*

1234. A solid is generated when the region in the first quadrant bounded by the graph of $y = 1 + \sin^2 x$, the line $x = \pi/2$, the x-axis, and the y-axis is revolved about the x-axis. Its volume is found by evaluating which of the following integrals?

A) $\pi \int_0^1 (1 + \sin^4 x)\,dx$

B) $\pi \int_0^1 (1 + \sin^2 x)^2\,dx$

C) $\pi \int_0^{\pi/2} (1 + \sin^4 x)\,dx$

D) $\pi \int_0^{\pi/2} (1 + \sin^2 x)^2\,dx$

E) $\pi \int_0^{\pi/2} (1 + \sin^2 x)\,dx$

1235. The volume generated by revolving about the x-axis the region above the curve $y = x^3$, below the line $y = 1$, and between $x = 0$ and $x = 1$ is

A) $\frac{\pi}{42}$ B) 0.143π C) $\frac{\pi}{7}$ D) 0.857π E) $\frac{64\pi}{7}$

1236. Find the distance traveled (to three decimal places) from $t = 1$ to $t = 5$ seconds, for a particle whose velocity is given by $v(t) = t + \ln t$.

A) 6.000 B) 1.609 C) 16.047 D) 0.800 E) 148.413

1237. A region is enclosed by the graphs of the line $y = 2$ and the parabola $y = 6 - x^2$. Find the volume of the solid generated when this region is revolved about the x-axis.

A) 76.8 B) 107.2 C) 167.6 D) 183.3 E) 241.3

1238. Find the area of the region above the x-axis and beneath one arch of the graph of $y = \frac{1}{2} + \sin x$.

A) $\frac{2}{3}\pi + \sqrt{3}$

B) $\frac{2}{3}\pi + 1$

C) $\sqrt{3} - \frac{1}{3}\pi$

D) $\sqrt{3} + \frac{4}{3}\pi$

E) $\frac{7}{12}\pi + \frac{1}{2}\sqrt{3} + 1$

Practical application is found by not looking for it, and one can say that the whole progress of civilization rests on that principle. —*Jacques Hadamard*

1239. The velocity of a particle moving along the x-axis is given by $v(t) = t\sin(t^2)$. Find the total distance traveled from $t = 0$ to $t = 3$.

A) 1.0 B) 1.5 C) 2.0 D) 2.5 E) 3.0

1240. Let $f(x)$ be a differentiable function whose domain is the closed interval $[0, 5]$, and let $F(x) = \int_0^x f(t)\, dt$. If $F(5) = 10$, which of the following must be true?

I. $F(x) = 2$ for some value of x in $[0, 5]$.
II. $f(x) = 2$ for some value of x in $[0, 5]$.
III. $f'(x) = 2$ for some value of x in $[0, 5]$.

A) I only B) II only C) III only D) I and II E) I, II, and III

1241. The base of a solid is the region in the xy-plane beneath the curve $y = \sin(kx)$ and above the x-axis for $0 \le x \le \frac{\pi}{2k}$. Each of the solid's cross-sections perpendicular to the x-axis has shape of a rectangle with height $\cos^2(kx)$. If the volume of the solid is 1 cubic unit, find the value of k. (Assume $k > 0$.)

A) 3 B) 3π C) $\dfrac{1}{3\pi}$ D) $\dfrac{\pi}{3}$ E) $\dfrac{1}{3}$

1242. The average value of $g(x) = (x - 3)^2$ in the interval $[1, 3]$ is

A) 2 B) $\frac{2}{3}$ C) $\frac{4}{3}$ D) $\frac{8}{3}$ E) None of these

---

The traditional mathematics professor of the popular legend is absentminded. He usually appears in public with a lost umbrella in each hand. He prefers to face the blackboard and to turn his back to the class. He writes a, he says b, he means c; but it should be d. Some of his sayings are handed down from generation to generation:
"In order to solve this differential equation you look at it till a solution occurs to you."
"This principle is so perfectly general that no particular application of it is possible."
"Geometry is the science of correct reasoning on incorrect figures."
"My method to overcome a difficulty is to go round it."
"What is the difference between method and device? A method is a device which you used twice."
—*George Polya*

A.P. Calculus Test Five
Section One
Multiple-Choice
No Calculators

Time—35 minutes

Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where C is the number of correct responses and I is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1. $\displaystyle\int_{\pi/4}^{\pi/2} \sin^3\alpha\cos\alpha\ d\alpha =$

A) $\frac{3}{16}$

B) $\frac{1}{8}$

C) $-\frac{1}{8}$

D) $-\frac{3}{16}$

E) $\frac{3}{4}$

---

2. If the distance of a particle from the origin on a line is given by $x(t) = 3 + (t-2)^4$, then the number of times the particle reverses direction is

A) 0

B) 1

C) 2

D) 3

E) None of these

---

3. $\displaystyle\int \tan x\ dx =$

A) $-\ln|\sec x| + C$

B) $\sec^2 x + C$

C) $\ln|\sin x| + C$

D) $\sec x + C$

E) $-\ln|\cos x| + C$

4. Solve the differential equation $\dfrac{dy}{dx} = y$ with the initial condition that $y(0) = 1$. From your solution, find the value of $y(e)$.

A) e^e

B) e

C) $e - 1$

D) $e^e - e$

E) e^2

---

5. The average value of $p(x) = \dfrac{1}{x}$ from $x = 1$ to $x = e$ is

A) $\dfrac{1}{e+1}$

B) $\dfrac{1}{1-e}$

C) $e - 1$

D) $1 - \dfrac{1}{e}$

E) $\dfrac{1}{e-1}$

---

6. The volume of a solid generated by revolving the region enclosed by the curve $y = 3x^2$ and the line $y = 6x$ about the x-axis is represented by

A) $\pi \displaystyle\int_0^3 (6x - 3x^2)^2 \, dx$

B) $\pi \displaystyle\int_0^2 (6x - 3x^2)^2 \, dx$

C) $\pi \displaystyle\int_0^2 (9x^4 - 36x^2) \, dx$

D) $\pi \displaystyle\int_0^2 (36x^2 - 9x^4) \, dx$

E) $\pi \displaystyle\int_0^2 (6x - 3x^2) \, dx$

7. A region in the plane is bounded by $y = \dfrac{1}{\sqrt{x}}$, the x-axis, the line $x = m$, and the line $x = 2m$, where $m > 0$. A solid is formed by revolving the region about the x-axis. The volume of this solid

A) is independent of m.

B) increases as m increases.

C) decreases as m decreases.

D) increases until $m = \frac{1}{2}$, then decreases.

E) cannot be found with the information given.

---

8. If the graph of $y = f(x)$ contains the point $(0, 1)$, and if $\dfrac{dy}{dx} = \dfrac{x\sin(x^2)}{y}$, then $f(x) =$

A) $\sqrt{2 - \cos(x^2)}$

B) $\sqrt{2} - \cos(x^2)$

C) $2 - \cos(x^2)$

D) $\cos(x^2)$

E) $\sqrt{2 - \cos x}$

---

9. $\displaystyle\lim_{h \to 0} \left(\frac{\tan(x + h) - \tan x}{h} \right) =$

A) $\sec x$

B) $-\sec x$

C) $\sec^2 x$

D) $-\sec^2 x$

E) does not exist

10. Given the differential equation $\dfrac{dy}{dx} = x + y$ with initial condition $y(0) = 2$, approximate $y(1)$ using Euler's method with a step size of 0.5.

A) 3

B) $\frac{7}{2}$

C) $\frac{15}{4}$

D) $\frac{19}{4}$

E) $\frac{21}{4}$

---

11. The base of a solid is a right triangle whose perpendicular sides have lengths 6 and 4. Each plane section of the solid perpendicular to the side of length 6 is a semicircle whose diameter lies in the plane of the triangle. The volume of the solid is

A) 2π units^3.

B) 4π units^3.

C) 8π units^3.

D) 16π units^3.

E) 24π units^3.

---

12. Which of the following expressions represents the length of the curve $y = e^{-x^2}$ for x from 0 to 2 ?

A) $\displaystyle\int_0^2 \sqrt{1 + e^{-2x^2}}\, dx$

B) $\displaystyle\int_0^2 \sqrt{1 + 4x^2 e^{-2x^2}}\, dx$

C) $\displaystyle\int_0^2 \sqrt{1 - e^{-2x^2}}\, dx$

D) $\displaystyle\int_0^2 \sqrt{1 + 2xe^{-2x^2}}\, dx$

E) $\displaystyle\pi \int_0^2 e^{-2x^2}\, dx$

13. If $f(x) = \displaystyle\int_2^{\sin x} \sqrt{1+t^2}\, dt$, then $f'(x) =$

A) $(1+x^2)^{3/2}$

B) $(\cos x)\sqrt{1+\sin x}$

C) $\sqrt{1+\sin^2 x}$

D) $(\cos x)\sqrt{1+\sin^2 x}$

E) $(\cos x)(1+\sin^2 x)^{3/2}$

---

14. For what value of x is the line tangent to $y = x^2$ parallel to the line tangent to $y = \sqrt{x}$?

A) 0

B) $\dfrac{1}{4\sqrt[3]{4}}$

C) $\dfrac{1}{2}$

D) $\dfrac{1}{2\sqrt[3]{2}}$

E) 1

---

15. An antiderivative of $(x^2+1)^2$ is

A) $\frac{1}{3}(x^2+1)^3 + C$

B) $\frac{1}{5}x^5 + x + C$

C) $\frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C$

D) $\frac{1}{6x}(x^2+1) + C$

E) $4x(x^2+1) + C$

A.P. Calculus Test Five
Section Two
Free-Response
Calculators Allowed

Time—45 minutes

Number of Questions—3

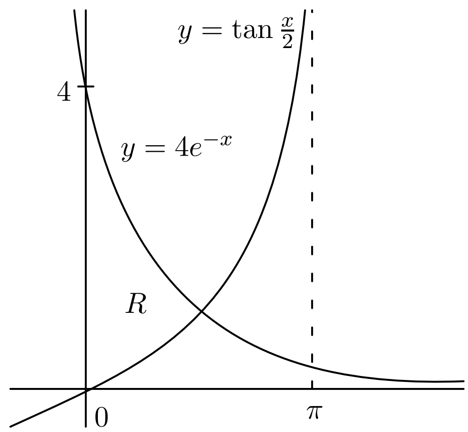
Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. Let R be the region in the first quadrant enclosed by the graphs of $y = 4e^{-x}$, $y = \tan\left(\frac{x}{2}\right)$, and the y-axis, as shown in the figure above.



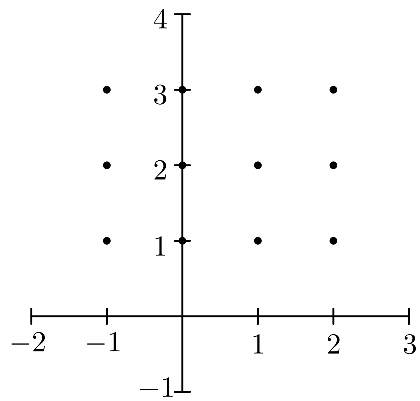
a) Find the area of region R.

b) Find the volume of the solid generated when the region R is revolved about the x-axis.

c) The region R is the base of a solid. For this solid, each cross-section perpendicular to the x-axis is a semicircle. Find the volume of this solid.

---

2. Consider the differential equation $\frac{dy}{dx} = \frac{xy}{2}$ with initial condition $y(0) = 2$.



a) Sketch the slope field for the given differential equation at the twelve points indicated.

b) Sketch the solution curve that satisfies the initial condition $y(0) = 2$ on the slope field above.

c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $y(0) = 2$. Then use your solution to find the exact value of $y(2)$.

---

3. A particle moves on the y-axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$.

a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?

b) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at $t = 1.5$? Why or why not?

c) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.

d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

CHAPTER 6

TECHNIQUES of INTEGRATION

6.1 A Part, And Yet, Apart...

FIND ANTIDERIVATIVES OF THE FOLLOWING BY PARTS.

1243. $\int x \ln x \, dx$

1244. $\int \arctan x \, dx$

1245. $\int 2xe^x \, dx$

1246. $\int 3\theta \sin(2\theta) \, d\theta$

1247. $\int \arcsin(2x) \, dx$

1248. $\int \ln(4x) \, dx$

1249. $\int 2^x x \, dx$

1250. $\int (x^2 - 5x)e^x \, dx$

1251. $\int e^x \sin x \, dx$

1252. $\int x \sec^2 x \, dx$

SOLVE THE DIFFERENTIAL EQUATIONS.

1253. $\dfrac{dy}{dx} = x^2 e^{4x}$

1254. $\dfrac{dy}{dx} = x^2 \ln x$

1255. $\dfrac{dy}{d\theta} = \sin \sqrt{\theta}$

1256. $\dfrac{dy}{d\theta} = \theta \sec \theta \tan \theta$

SOLVE THE FOLLOWING.

1257. Find the area bounded by the curve $y = \ln x$ and the lines $y = 1$ and $x = e^2$.

1258. Find the area bounded by the curve $y = \ln(x + 3)$, the line $y = 1$, and the y-axis.

1259. Find the area of the region bounded entirely by the curves $y = \ln x$ and $y = (\ln x)^2$.

1260. Find the area between the curves $y = 5e^x$ and $y = 4x^3 + \ln x$ over the interval $[1, 2]$.

1261. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

1262. Find the average value of $y = 2e^{-x} \cos x$ over the interval $[0, 2\pi]$.

1263. Graph the function $f(x) = x \sin x$ in the window $0 \leq x \leq 3\pi$, $-5 \leq y \leq 10$, using an x-scale of π and a y-scale of 5. Find the area of the region between f and the x-axis for

a) $0 \leq x \leq \pi$

b) $\pi \leq x \leq 2\pi$

c) $2\pi \leq x \leq 3\pi$

d) What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \leq x \leq (n + 1)\pi$ for any nonnegative integer n?

Advertising may be described as the science of arresting human intelligence long enough to get money from it. —*Stephen Leacock*

6.2 Partial Fractions

EVALUATE THE FOLLOWING BY PARTIAL FRACTIONS.

1264. $\displaystyle\int \frac{1}{x^2-9}\,dx$

1265. $\displaystyle\int \frac{1}{1-x^2}\,dx$

1266. $\displaystyle\int \frac{3x-2}{x^2-9}\,dx$

1267. $\displaystyle\int \frac{x^2}{x^2-2x-15}\,dx$

1268. $\displaystyle\int \frac{3x^2-2x+1}{9x^3-x}\,dx$

1269. $\displaystyle\int_{1/2}^{1} \frac{x+4}{x^2+x}\,dx$

1270. $\displaystyle\int \frac{3x-2}{x^3-3x^2-10x}\,dx$

1271. $\displaystyle\int \frac{2}{x^2(x-5)}\,dx$

1272. $\displaystyle\int \frac{1}{x^2(x-5)^2}\,dx$

1273. $\displaystyle\int_{4}^{8} \frac{x}{x^2-2x-3}\,dx$

SOLVE THE FOLLOWING INITIAL VALUE PROBLEMS.

1274. $\displaystyle\frac{dy}{dx} = (y^2-y)e^x, \quad y(0)=2$

1275. $\displaystyle\frac{dy}{d\theta} = (y+1)^2\sin\theta, \quad y(\pi/2)=0$

1276. $\displaystyle\frac{dy}{dx} = \frac{1}{x^2-3x+2}, \quad y(3)=0$

1277. $\displaystyle\frac{dy}{dt} = \frac{2y+2}{t^2+2t}, \quad y(1)=1$

1278. The growth of an animal population is governed by the equation $1000\dfrac{dP}{dt} = P(100-P)$, where $P(t)$ is the number of animals in the colony at time t. If the initial population was 200 animals, how many animals will there be when $t=20$?

1279. Consider the equation $\dfrac{dP}{dt} = 0.02P^2 - 0.08P$. Sketch the slope field for this equation for $0 \le t \le 50$ and $0 \le P \le 8$. Then sketch the solution curve corresponding to the initial condition $P(0)=1$. Finally, solve the equation using the given initial condition.

1280. Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population, such as a rumor, cultural fad, or news concerning a technological innovation. In a sufficiently large population, the rate of diffusion is assumed to be proportional to the number of people p who have the information times the number of people who do not. Thus, if N is the population size, then $\dfrac{dp}{dt} = kp(N-p)$. Suppose that t is in days, $k=\frac{1}{250}$, and two people start a rumor at time $t=0$ in a population of $N=1000$ people. Find $p(t)$ and determine how many days it will take for half the population to hear the rumor.

Biographical history, as taught in our public schools, is still largely a history of boneheads: ridiculous kings and queens, paranoid political leaders, compulsive voyagers, ignorant generals – the flotsam and jetsam of historical currents. The men who radically altered history, the great scientists and mathematicians, are seldom mentioned, if at all. —*Martin Gardner*

6.3 Trigonometric Substitution

Evaluate the following by using a trig substitution.

1281. $\displaystyle\int \frac{3}{\sqrt{1+9x^2}}\,dx$

1282. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

1283. $\displaystyle\int \frac{\sqrt{x^2-9}}{x^2}\,dx$

1284. $\displaystyle\int \frac{1}{x^2\sqrt{4x^2-9}}\,dx$

1285. $\displaystyle\int \frac{1}{(1-x^2)^{3/2}}\,dx$

1286. $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$

Solve the following initial value problems.

1287. $x\dfrac{dy}{dx} = \sqrt{x^2-4}, \quad x \geq 2,\ y(2) = 0$

1288. $\sqrt{x^2-9}\dfrac{dy}{dx} = 1, \quad x > 3,\ y(5) = \ln 3$

1289. $(x^2+4)\dfrac{dy}{dx} = 3, \quad y(2) = 0$

1290. $(x^2+1)^2\dfrac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Solve the following problems.

1291. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \frac{1}{3}\sqrt{9-x^2}$.

1292. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant bounded by the coordinate axes, the curve $y = \dfrac{2}{1+x^2}$, and the line $x = 1$.

1293. Consider the region bounded by the graphs of $y = e^x$, $y = 0$, $x = 1$, and $x = 2$. Find the value of d for which the line $x = d$ divides the area of the region in a $2:1$ ratio.

1294. Find the volume of the solids formed by revolving the following curves about the x-axis over the given interval.

a) $y = xe^{x/2}$ over $[0, 1]$

b) $y = \sqrt{xe^x}$ over $[1, 2]$

c) $y = \ln x$ over $[1, 2]$

d) $y = \sqrt{1+x}$ over $[1, 5]$

As I understand it, the first time Gabriel García Márquez opened Kafka's *The Metamorphosis*, he was a teenager, reclining on a couch. Upon reading

> As Gregor Samsa awoke one morning from uneasy dreams he found himself transformed in his bed into a gigantic insect...

García Márquez fell off his couch, astonished by the revelation that you were *allowed* to write like that! It has happened to me often, and surely a similar thing happens to all mathematicians, that upon hearing of someone's new idea, or new construction, I have, like García Márquez, fallen off my (figurative) couch, thinking in amazement, "I didn't realize we were *allowed* to do that!" —*Barry Mazur*

6.4 Four Integral Problems

Solve each of the following.

1295. *Guess* which of the following two integrals will be larger. Explain your reasoning.

$$\int_0^4 x\sqrt{16-x^2}\ dx \qquad \int_0^4 \sqrt{16-x^2}\ dx$$

Then *compute* which of the two integrals is actually larger.

1296. Show that the region enclosed by the graph of the parabola

$$f(x) = \frac{2}{a^2}x - \frac{1}{a^3}x^2, \quad a > 0$$

and the x-axis has an area that is independent of the value of a. How large is this area? What curve is determined by the vertices of all these parabolas?

1297 (Calculator). Let R be the region bounded by $f(x) = e^{\sin 2x}\cos 2x$ and $g(x) = x^2$.

a) Find the area of R. Your answer must include an antiderivative.

b) Find the volume of the solid formed by revolving R about the line $x = -1$.

c) Set up an integral that represents the volume of the solid whose base is R and the cross-sections perpendicular to the x-axis are squares. Use your calculator to evaluate the integral.

1298. Let f and g be continuous and differentiable functions satisfying the given conditions for some real number B:

I. $\displaystyle\int_1^3 f(x+2)\ dx = 3B$

II. The average value of f in the interval $[1,3]$ is $2B$

III. $\displaystyle\int_{-4}^x g(t)\ dt = f(x) + 3x$

IV. $g(x) = 4B + f'(x)$

a) Find $\displaystyle\int_1^5 f(x)\ dx$ in terms of B.

b) Find B.

"Alice laughed: 'There's no use trying,' she said; 'one can't believe impossible things.'
'I daresay you haven't had much practice,' said the Queen. 'When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast.' " —*Lewis Carroll,* Through the Looking Glass

6.5 L'Hôpital's Rule

Evaluate each of the following limits.

1299. $\displaystyle\lim_{x\to 0} \frac{e^x}{x}$

1300. $\displaystyle\lim_{x\to\infty} \frac{e^{4x}}{5x}$

1301. $\displaystyle\lim_{x\to\infty} \frac{(x+5)^2}{e^{3x}}$

1302. $\displaystyle\lim_{x\to 0} \frac{e^{3x}-2^x}{3x}$

1303. $\displaystyle\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

1304. $\displaystyle\lim_{x\to\infty} 3x\ln\left(1+\frac{1}{x}\right)$

1305. $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3}$

1306. $\displaystyle\lim_{x\to 2} \frac{\ln(3-x)}{1-e^{x/2-1}}$

1307. $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{2x}$

1308. $\displaystyle\lim_{x\to 0} \frac{\sin(8x)}{6x}$

1309. $\displaystyle\lim_{x\to 0} \frac{\tan(3x)}{2x}$

1310. $\displaystyle\lim_{x\to\pi/2} \frac{1-\sin x}{x-\pi/2}$

1311. $\displaystyle\lim_{x\to 1} \frac{3x^2-5x+2}{x-1}$

1312. $\displaystyle\lim_{x\to\infty} \frac{3x^2-5x+2}{x-1}$

1313. $\displaystyle\lim_{x\to 0^+} x^x$

1314. $\displaystyle\lim_{x\to\infty} (\ln x)^{1/x}$

1315. $\displaystyle\lim_{x\to\infty} (1+2x)^{1/(2\ln x)}$

1316. $\displaystyle\lim_{x\to 1} (x^2-2x+1)^{x-1}$

1317. $\displaystyle\lim_{x\to\infty} x^{1/x}$

1318. $\displaystyle\lim_{x\to 0^+} (1+x)^{1/x}$

1319. $\displaystyle\lim_{x\to 1} x^{1/(x-1)}$

1320. $\displaystyle\lim_{x\to 0^+} (\sin x)^x$

1321. $\displaystyle\lim_{x\to 0^+} (\sin x)^{\tan x}$

1322. $\displaystyle\lim_{x\to\infty} x^2e^{-x}$

1323. $\displaystyle\lim_{x\to\infty} \int_x^{2x} \frac{1}{t}\,dt$

1324. $\displaystyle\lim_{x\to\infty} \frac{1}{x\ln x}\int_1^x \ln t\,dt$

1325. $\displaystyle\lim_{x\to 0} \frac{\cos x-1}{e^x-x-1}$

1326. $\displaystyle\lim_{x\to\infty} \frac{e^x+x^2}{e^x-x}$

Solve the following problems.

1327. Find the value of c that makes the function below continuous at $x=0$.

$$f(x)=\begin{cases}\dfrac{9x-3\sin(3x)}{5x^3} & x\neq 0\\ c & x=0\end{cases}$$

1328. Estimate the value of $\displaystyle\lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ by graphing. Then confirm your answer by using l'Hôpital's rule.

1329. Let $f(x)=\dfrac{1-\cos(x^6)}{x^{12}}$.

a) Graph f on the standard window ($-10\le x\le 10$, $-10\le y\le 10$) and use the graph to determine $\displaystyle\lim_{x\to 0} f(x)$.

b) Now graph f on the window $-1\le x\le 1$, $-0.5\le y\le 1$. What does the limit appear to be now?

c) What does this indicate about finding limits using a graphing calculator?

My goal in life is to have one of my quotes in here. —*Jesse Smith*

6.6 Improper Integrals!

Let b be a number greater than 1. Evaluate the following integrals in terms of b, then find the limit as $b \to \infty$. What do these answers mean in terms of area under the curve?

1330. $\int_1^b \frac{1}{x^2}\,dx$

1331. $\int_1^b \frac{1}{x^3}\,dx$

1332. $\int_1^b \frac{1}{x^{1/2}}\,dx$

1333. $\int_1^b \frac{1}{x^{1/3}}\,dx$

1334. $\int_1^b \frac{1}{x^{3/2}}\,dx$

1335. $\int_1^b \frac{1}{x}\,dx$

Let b be a number between 0 and 1. Evaluate the following integrals in terms of b, then find the limit as $b \to 0^+$. What do these answers mean in terms of area under the curve?

1336. $\int_b^1 \frac{1}{x^2}\,dx$

1337. $\int_b^1 \frac{1}{x^3}\,dx$

1338. $\int_b^1 \frac{1}{x^{1/2}}\,dx$

1339. $\int_b^1 \frac{1}{x^{1/3}}\,dx$

1340. $\int_b^1 \frac{1}{x^{3/2}}\,dx$

1341. $\int_b^1 \frac{1}{x}\,dx$

Evaluate the following.

1342. $\int_0^\infty \frac{1}{t^2+9}\,dt$

1343. $\int_0^\infty e^{-x}\,dx$

1344. $\int_{-\infty}^0 \frac{1}{(t-1)^2}\,dt$

1345. $\int_0^\infty \cos\theta\,d\theta$

1346. $\int_0^\infty xe^{-x^2}\,dx$

1347. $\int_0^{\pi/2} \tan\theta\,d\theta$

1348. $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

1349. $\int_0^1 x\ln x\,dx$

1350. $\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

1351. $\int_2^\infty \frac{2}{t^2-t}\,dt$

1352. $\int_0^1 \frac{4t}{\sqrt{1-t^4}}\,dt$

1353. $\int_0^\infty \frac{16\arctan x}{1+x^2}\,dx$

1354. $\int_{-1}^\infty \frac{1}{x^2+5x+6}\,dx$

1355. $\int_1^\infty x^{-0.99}\,dx$

1356. $\int_1^\infty x^{-1.01}\,dx$

1357. Consider the region in the first quadrant between the curve $y = e^{-x}$ and the x-axis. Find the area of the region; the volume of the solid formed when the region is revolved about the y-axis; and the volume of the solid formed when the region is revolved about the x-axis.

1358. Let R be the region between the curves $y = 1/x$ and $y = 1/(x+1)$, to the right of the line $x = 1$. Find the area of this region if it is finite.

1359. A patient is given an injection of imitrex, a medicine to relieve migraine headaches, at a rate of $r(t) = 2te^{-2t}$ ml/sec, where t is the number of seconds since the injection started. Estimate the total quantity of imitrex injected.

You need to put one of my quotes in here, because it's my goal in life. —*Justin Easley*

6.7 The Art of Integration

1360 (AP). If the substitution $u = x/2$ is made, the integral $\displaystyle\int_2^4 \frac{1-(x/2)^2}{x}\,dx =$

A) $\displaystyle\int_1^2 \frac{1-u^2}{u}\,du$ B) $\displaystyle\int_2^4 \frac{1-u^2}{u}\,du$ C) $\displaystyle\int_2^4 \frac{1-u^2}{4u}\,du$ D) $\displaystyle\int_2^4 \frac{1-u^2}{2u}\,du$ E) $\displaystyle\int_1^2 \frac{1-u^2}{2u}\,du$

1361. *Partial Fractions Versus Trig Substitution*

a) Graph the function $f(x) = \dfrac{1}{x^2-4}$ on your paper.

b) Is the definite integral $\displaystyle\int_{-1}^1 \frac{dx}{x^2-4}$ negative or positive? Justify your answer with reference to your graph.

c) Compute the integral in part (b) by using partial fractions.

d) A Georgia Tech calculus student suggests instead to use the substitution $x = 2\sec\theta$. Compute the integral in this way, or describe why this substitution fails.

1362 (AP). If $\displaystyle\int f(x)\sin x\,dx = -f(x)\cos x + \int 3x^2\cos x\,dx$, then $f(x)$ could be

A) $3x^2$ B) x^3 C) $-x^3$ D) $\sin x$ E) $\cos x$

1363. Justin and Jonathan are having an argument as to the value of $\int \sec^2 x \tan x\,dx$. Justin makes the substitution $u = \sec x$ and gets the answer $\frac{1}{2}\sec^2 x$, whereas Jonathan makes the substitution $u = \tan x$ and gets the answer $\frac{1}{2}\tan^2 x$. Please get them to stop arguing by explaining to them why their antiderivatives are both acceptable.

1364. Determine which of the following converge (are finite) and diverge (are infinite) by comparing the integrands to other known integrals.

A) $\displaystyle\int_1^\infty \frac{dx}{1+x^4}$ B) $\displaystyle\int_1^\infty \frac{x\,dx}{\sqrt{1+x^3}}$ C) $\displaystyle\int_0^\infty e^{-x^2}\,dx$ D) $\displaystyle\int_1^\infty \frac{\sin x}{x^2}\,dx$

Only an idiot could believe that science requires martyrdom – that may be necessary in religion, but in time a scientific result will establish itself. —*David Hilbert*

1365. A function G is defined by $G(x) = \int_0^x \sqrt{1+t^2}\, dt$ for all real numbers x. Determine whether the following statements are true or false. Justify your answers.

A) G is continuous at $x = 0$.

B) $G(3) > G(1)$.

C) $G'(2\sqrt{2}) = 3$.

D) The graph of G has a horizontal tangent at $x = 0$.

E) The graph of G has an inflection point at $(0, 0)$.

1366. Consider the following table of values for the differentiable function f.

x	1.0	1.2	1.4	1.6	1.8
$f(x)$	5.0	3.5	2.6	2.0	1.5

a) Estimate $f'(1.4)$.

b) Give an equation for the tangent line to the graph of f at $x = 1.4$.

c) Is $f''(x)$ positive, negative, or zero? Explain how you determine this.

d) Using the data in the table, find a midpoint approximation with 2 equal subintervals for $\int_{1.0}^{1.8} f(x)\, dx$.

1367. The gamma function $\Gamma(x)$ is defined for all $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

a) Evaluate $\Gamma(1)$.

b) For $x > 1$, show that $\Gamma(x) = x\Gamma(x-1)$. Assume that all these improper integrals exist. *Hint:* Use integration by parts.

c) Use parts (a) and (b) to find $\Gamma(2)$, $\Gamma(3)$, and $\Gamma(4)$. What is the pattern?

d) One of the few values of $\Gamma(x)$ for noninteger x that can be evaluated exactly is $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2}\, dt$, whose value is $\sqrt{\pi}$. Explain why $\Gamma(\frac{1}{2})$ converges.

e) Try to evaluate $\Gamma(\frac{1}{2})$ on your calculator.

If you ask mathematicians what they do, you always get the same answer. They think. They think about difficult and unusual problems. They do not think about ordinary problems: they just write down the answers to those. —*M. Egrafov*

6.8 Functions Defined By Integrals

SOLVE THE FOLLOWING PROBLEMS.

1368 (AP). $\lim_{x\to 0} \dfrac{\int_1^{1+x} \sqrt{t^2+8}}{x}$ is

A) 0   B) 1   C) 3   D) $2\sqrt{2}$ E) nonexistent

1369. Find the derivatives of the functions defined by the following integrals.

a) $\displaystyle\int_0^x \frac{\sin t}{t}\, dt$

b) $\displaystyle\int_0^x e^{-t^2}\, dt$

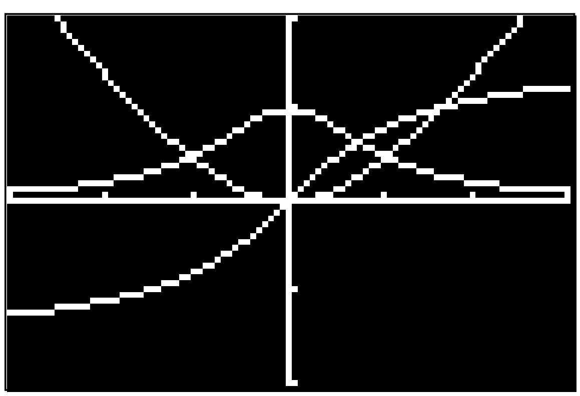
c) $\displaystyle\int_1^{\cos t} \frac{1}{t}\, dt$

d) $\displaystyle\int_0^1 e^{\tan^2 t}\, dt$

e) $\displaystyle\int_1^{\ln x} e^{t^2}\, dt$

f) $\displaystyle\int_x^{x^2} \frac{1}{2t}\, dt$

1370. The graphs of three functions appear in the figure below. Idenitfy which is $f(x)$, which is $f'(x)$, and which is $\displaystyle\int_0^x f(t)\, dt$.



Science without religion is lame; religion without science is blind. —*Albert Einstein*

1371. Let $F(x) = \int_0^x f(t)\,dt$ where f is the function graphed below.

a) Evaluate $\int_0^2 f(t)\,dt$, $\int_0^4 f(t)\,dt$, $\int_2^4 f(t)\,dt$, $\int_5^{10} f(t)\,dt$, and $\int_1^7 f(t)\,dt$.

b) Evaluate $F(0)$, $F(2)$, $F(5)$, and $F(7)$.

c) Find an analytic expression for $f(x)$.

d) Find an analytic expression for $F(x)$.

e) Sketch the graphs of f and F on the same axes over the interval $[0, 10]$.

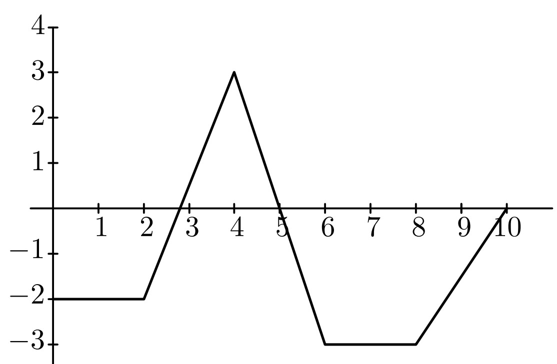
f) Where does F have local maxima on the interval $[0, 10]$?

g) On which subintervals of $[0, 10]$, if any, is F decreasing?

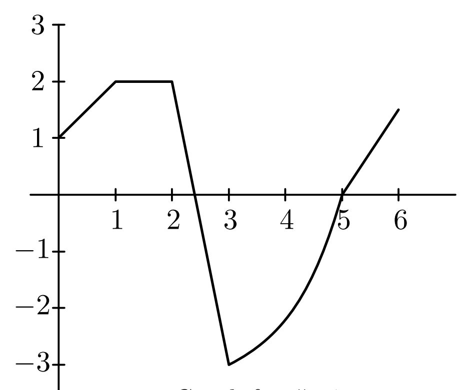
h) On which subintervals of $[0, 10]$, if any, is F increasing?

i) On which subintervals of $[0, 10]$, if any, is the graph of F concave up?

j) On which subintervals of $[0, 10]$, if any, is the graph of F concave down?



Graph for #1371



Graph for #1372

1372. Let $F(x) = \int_1^x f(t)\,dt$, where f is the function graphed above.

a) Suppose $\int_0^5 f(t)\,dt = -\frac{2}{3}$. What is $F(5)$?

b) Show that F has exactly one zero between 3 and 4.

c) Find the equation of the tangent line to the graph of F at the point $(3, F(3))$. *Hint:* What is $F'(3)$?

d) Use the equation found in part (c) to approximate the zero of F between 3 and 4.

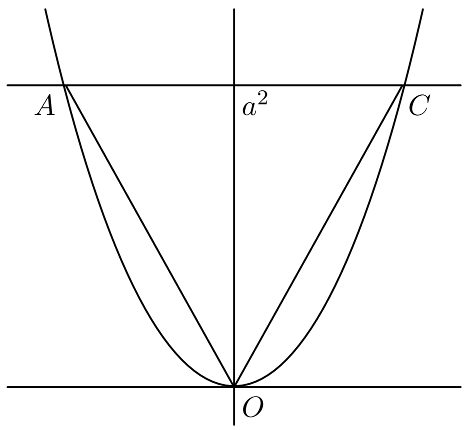
Everything should be made as simple as possible, but not simpler. —*Albert Einstein*

6.9 Sample A.P. Problems on Techniques of Integration

1373. Let $A(t)$ be the area of the region in the first quadrant enclosed by the coordinate axes, the curve $y = e^{-x}$, and the line $x = t > 0$. Let $V(t)$ be the volume of the solid generated by revolving the region about the x-axis. Find the following limits.

a) $\lim_{t\to\infty} A(t)$ b) $\lim_{t\to\infty} \dfrac{V(t)}{A(t)}$ c) $\lim_{t\to 0^+} \dfrac{V(t)}{A(t)}$

1374. The figure below shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.



1375. Find the area of the region enclosed by the curves $y = x^2$, $y = (x^2 + x + 1)e^{-x}$, and the y-axis.

1376. Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentration of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of the product may be given by the separable differential equation

$$\frac{dx}{dt} = k(a - x)(b - x)$$

where k is a constant for the reaction. Assuming that $x = 0$ when $t = 0$, solve this equation to obtain a relation between x and t.

a) if $a = b$ and

b) if $a \neq b$.

1377. For what value of a does

$$\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x}\right)\, dx$$

converge? To what value does it converge?

1378. Let R be the region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving the R about

a) the x-axis

b) the line $y = 1$

c) the y-axis

d) the line $x = 1$.

1379. The region between the x-axis and the curve

$$y = \begin{cases} 0 & x = 0 \\ x \ln x & 0 < x \leq 2 \end{cases}$$

is revolved around the x-axis to generate a solid.

a) Show that y is continuous at $x = 0$.

b) Find the volume of the solid.

1380. A single infected individual enters a community of n susceptible individuals. Let x be the number of newly infected individuals at time t. The common epidemic model assumes that the disease spreads at a rate proportional to the product of the total number infected and the number not yet infected. Thus, the spread is modeled by

$$\frac{dx}{dt} = k(x+1)(n-x).$$

a) Find $x(t)$, the solution to the differential equation, in terms of k and n.

b) If an infected person enters a community of 1500 susceptible individuals, and 100 are infected after 15 days, how many days will it take for 1000 people to be infected?

1381 (1996BC, Calculator). Consider the graph of the function h given by $h(x) = e^{-x^2}$ for $0 \leq x < \infty$.

a) Let R be the unbounded region in the first quadrant below the graph of h. Find the volume of the solid generated when R is revolved about the y-axis.

b) A rectangle has one vertex at the origin, one on the x-axis at $x = w$, one on the y-axis and another on the graph of h in the first quadrant. Let $A(w)$ be the area of the rectangle. Show that $A(w)$ has its maximum value when w is the x-coordinate of the point of inflection of the graph of h.

1382. Find the area under the arch of the ellipse $y = \frac{4}{25}\sqrt{25 - x^2}$ and above the x-axis.

1383 (1969BC). A region R has $y = 1 + \sin\left(\dfrac{\pi x}{2}\right)$ as its upper boundary, $y = \frac{1}{2}x$ as its lower boundary, and the y-axis as its left-hand boundary.

a) Sketch the region R.

b) Set up, but do not evaluate, an integral expression in terms of the single variable x, for

   i) the area of R;

   ii) the volume of the solid formed by revolving R about the x-axis;

   iii) and the total perimeter of R.

1384 (1980BC). Let R be the region enclosed by the graphs of $y = e^{-x}$, $x = k$ $(k > 0)$, and the coordinate axes.

a) Write an improper integral that represents the limit of the area of R as k increases without bound and find the value of the integral if it exists.

b) Find the volume, in terms of k, of the solid generated by rotating R around the y-axis.

c) Find the volume, in terms of k, of the solid whose base is R and whose cross sections perpendicular to the x-axis are squares.

1385. When computing the internal energy of a crystal, Claude Garrod, in his book *Twentieth Century Physics* (published in 1984), states that the integral

$$\int_0^{\pi/2} \frac{\sin x}{e^{0.26 \sin x} - 1}\, dx$$

"cannot be evaluated analytically. However, it can easily be computed numerically using Simpson's rule. The result is 5.56."

a) Is the integral proper or improper? Why?

b) What is the limit of the integrand as $x \to 0^+$?

c) What does "cannot be evaluated analytically" mean?

d) Is it possible to use your calculator program to approximate the integral by Simpson's rule with $n = 6$? If so, approximate it to four decimal places; if not, why not?

More than 50% of all known mathematics was created during the past 50 years, and 50% of all mathematicians who have ever lived are alive today. —*Howard Eves*

6.10 Sample Multiple-Choice Problems on Techniques of Integration

1386. $\int x \sin x \, dx =$

A) $-x\cos x + C$

B) $-x\cos x - \sin x + C$

C) $-x\cos x + \sin x + C$

D) $\frac{1}{2}x^2 \sin x + C$

E) $-x\cos x - \cos x + C$

1387. $\int xe^{-x} \, dx =$

A) $e^{-x}(1-x) + C$

B) $\dfrac{e^{1-x}}{1-x} + C$

C) $-e^{-x}(x+1) + C$

D) $-\frac{1}{2}xe^{-x} + C$

E) $e^{-x}(x+1) + C$

1388. $\int \dfrac{\ln x}{x} \, dx =$

A) $\frac{1}{2}\ln x + C$

B) $\frac{1}{2}(\ln x)^2 + C$

C) $2\sqrt{\ln x} + C$

D) $\frac{1}{2}\ln x^2 + C$

E) None of these

1389. $\int \tan^{-1}(2x) \, dx =$

A) $\dfrac{2}{1+4x^2} + C$

B) $x\tan^{-1}(2x) + C$

C) $x\tan^{-1}(2x) + \frac{1}{4}\ln(1+4x^2) + C$

D) $x\tan^{-1}(2x) - \frac{1}{4}\ln(1+4x^2) + C$

E) None of these

1390. $\int \dfrac{x}{\sqrt{9-x^2}} \, dx =$

A) $-\frac{1}{2}\ln\sqrt{9-x^2} + C$

B) $\arcsin(\frac{x}{3}) + C$

C) $-\sqrt{9-x^2} + C$

D) $-\frac{1}{4}\sqrt{9-x^2} + C$

E) $2\sqrt{9-x^2} + C$

1391. $\int \tan x \, dx =$

A) $-\ln|\sec x| + C$

B) $\sec^2 x + C$

C) $\ln|\sin x| + C$

D) $\sec x + C$

E) $-\ln|\cos x| + C$

1392. $\int_0^1 \dfrac{e^x}{1+e^x} \, dx =$

A) $\ln 2$ B) e C) $1+e$ D) $-\ln 2$ E) $\ln\left(\dfrac{e+1}{2}\right)$

1393. $\int_0^{\pi/4} \tan^2\theta \, d\theta =$

A) $\frac{\pi}{4} - 1$ B) $\sqrt{2} - 1$ C) $\frac{\pi}{4} + 1$ D) $\frac{1}{3}$ E) $1 - \frac{\pi}{4}$

1394. $\lim\limits_{x\to\pi/2} \dfrac{\cos x}{x-\pi/2} =$

A) -1 B) 1 C) 0 D) ∞ E) None of these

1395. $\displaystyle\int_0^1 xe^x\,dx =$

A) 1 B) -1 C) $2-e$ D) $\frac{1}{2}e^2-e$ E) $e-1$

1396. $\displaystyle\int_1^e \ln x\,dx =$

A) $\frac{1}{2}$ B) $e-1$ C) $e+1$ D) 1 E) -1

1397. $\lim\limits_{x\to 0} x\sin(\frac{1}{x}) =$

A) 1 B) 0 C) ∞ D) -1 E) None of these

1398. Which of the following integrals is equal to $\frac{5}{4}$?

A) $\displaystyle\int_0^1 \frac{1}{x^{0.2}}\,dx$ B) $\displaystyle\int_0^1 \frac{1}{x^{0.5}}\,dx$ C) $\displaystyle\int_0^1 \frac{1}{x^{0.7}}\,dx$ D) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ E) None of these

1399. $\lim\limits_{h\to 0} \dfrac{-1+e^{-h}}{h} =$

A) 1 B) 0 C) -1 D) $\frac{1}{e}$ E) ∞

1400. The region bounded by $y=e^x$, $y=1$, and $x=2$ is revolved about the y-axis. The volume of this solid is

A) $2\pi(e^2-1)$ B) $\pi(e^2+1)$ C) $\pi(e^2-2)$ D) $2\pi(e^2-2)$ E) None of these

1401. The region bounded by $y=e^x$, $y=1$, and $x=2$ is revolved about the x-axis. The volume of this solid is

A) $\frac{\pi}{2}(e^4-4)$ B) $\pi(e^4-4)$ C) $\frac{\pi}{2}(e^4-5)$ D) $\frac{\pi}{2}(e^4-10)$ E) None of these

1402. The area in the first quadrant bounded by the curve $y=x^2$ and the line $y-x-2=0$ is

A) $\frac{3}{2}$ B) $\frac{2}{3}$ C) $\frac{7}{6}$ D) $\frac{10}{3}$ E) $\frac{9}{2}$

1403. $\dfrac{d}{dx}\displaystyle\int_3^{-5x^2} (7t-1)\,dt =$

A) $7x^2-1$ B) $-5x^2-1$ C) $-70x^3+10x$ D) $350x^3+10x$ E) None of these

Mathematical reality lies outside us, ... our function is to discover or observe it. —*G. H. Hardy*

A.P. Calculus BC Test Five
Section One
Multiple-Choice
No Calculators
Time—35 minutes
Number of Questions—15

The scoring for this section is determined by the formula

$$[C - (0.25 \times I)] \times 1.8$$

where C is the number of correct responses and I is the number of incorrect responses. An unanswered question earns zero points. The maximum possible points earned on this section is 27, which represents 50% of the total test score.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding choice on your answer sheet. Do not spend too much time on any one problem.

Good Luck!

NAME:

1. $\displaystyle\int x \sin x \; dx =$

A) $-x\cos x + C$

B) $-x\cos x - \sin x + C$

C) $-x\cos x + \sin x + C$

D) $\frac{1}{2}x^2 \sin x + C$

E) $-x\cos x - \cos x + C$

---

2. $\displaystyle\int_1^e \frac{\ln x}{x} \; dx =$

A) undefined

B) $\frac{1}{2}$

C) 2

D) $\frac{1}{2}(e-1)$

E) None of these

---

3. The area of the region bounded by the lines $x = 0$, $x = 2$, $y = 0$, and the curve $y = e^{x/2}$ is

A) $\frac{1}{2}(e-1)$

B) $e - 1$

C) $2(e-1)$

D) $2e - 1$

E) $2e$

4. $\lim_{h \to 0} \frac{-1+e^{-h}}{h} =$

A) 1

B) 0

C) -1

D) $\frac{1}{e}$

E) ∞

---

5. Evaluate $\int_1^\infty x^{-1/2}\ dx$.

A) 3

B) 2

C) 1

D) $\frac{1}{2}$

E) divergent

---

6. $\int \frac{1}{x^2+x}\ dx =$

A) $\frac{1}{2}\arctan\left(x+\frac{1}{2}\right)+C$

B) $\ln|x^2+x|+C$

C) $\ln\left|\frac{x+1}{x}\right|+C$

D) $\ln\left|\frac{x}{x+1}\right|+C$

E) None of these

7. $\displaystyle\int \frac{x}{x+2}\,dx =$

A) $x \ln|x+2| + C$

B) $x + 2\ln|x+2| + C$

C) $x - 2\ln|x+2| + C$

D) $x - \ln|x+2| + C$

E) $x - \arctan x + C$

---

8. A particle moves on the x-axis in such a way that its position at time t, for $t > 0$, is given by $x(t) = (\ln x)^2$. At what value of t does the velocity of the particle attain its maximum?

A) 1

B) $e^{1/2}$

C) e

D) $e^{3/2}$

E) e^2

---

9. The substitution of $x = \sin\theta$ in the integrand of $\displaystyle\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$ results in

A) $\displaystyle\int_0^{1/2} \frac{\sin^2\theta}{\cos\theta}\,d\theta$

B) $\displaystyle\int_0^{1/2} \sin^2\theta\,d\theta$

C) $\displaystyle\int_0^{\pi/6} \sin^2\theta\,d\theta$

D) $\displaystyle\int_0^{\pi/3} \sin^2\theta\,d\theta$

E) $\displaystyle\int_0^{1/2} \frac{\cos^2\theta}{\sin\theta}\,d\theta$

10. The area of the region in the first quadrant under the curve $y = \dfrac{1}{\sqrt{1-x^2}}$, bounded on the left by $x = \frac{1}{2}$, and on the right by $x = 1$ is

A) ∞

B) π

C) $\pi/2$

D) $\pi/3$

E) None of these

---

11. The length of the curve $y = \displaystyle\int_0^x \sqrt{\frac{t}{3}}\, dt$ from $x = 0$ to $x = 9$ is

A) 16.

B) 14.

C) $\frac{31}{3}$.

D) $9\sqrt{3}$.

E) $\frac{14}{3}$.

---

12. Evaluate $\displaystyle\int_{-5}^{5} \sqrt{25 - x^2}\, dx$.

A) 0

B) 5

C) $25\pi/2$

D) 25π

E) 50π

13. Consider the function g defined by $g(x) = \int_1^x (t^3 - 3t^2 + 2t)\, dt$. The number of relative extrema of g is

A) 1.

B) 2.

C) 3.

D) 4.

E) more than 4.

---

14. The function $t(x) = 2^x - \dfrac{|x-3|}{x-3}$ has

A) a removable discontinuity at $x = 3$.

B) an infinite discontinuity at $x = 3$.

C) a jump discontinuity at $x = 3$.

D) no discontinuities.

E) a removable discontinuity at $x = 0$ and an infinite discontinuity at $x = 3$.

---

15. Find the values of c so that the function

$$h(x) = \begin{cases} c^2 - x^2 & x < 2 \\ x + c & x \geq 2 \end{cases}$$

is continuous everywhere.

A) $-3, -2$

B) $2, 3$

C) $-2, 3$

D) $-3, 2$

E) There are no such values.

A.P. Calculus BC Test Five
Section Two
Free-Response
Calculators Allowed

Time—45 minutes

Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\, dx$ may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

1. Let f be a differentiable function defined for all $x \geq 0$ such that $f(0) = 5$ and $f(3) = -1$. Suppose that for any number $b > 0$ the average value of $f(x)$ on the interval $0 \leq x \leq b$ is $\dfrac{f(0) + f(b)}{2}$.

a) Find $\displaystyle\int_0^3 f(x)\,dx$.

b) Prove that $f'(x) = \dfrac{f(x) - 5}{x}$ for all $x > 0$.

c) Using part (b), find $f(x)$.

---

2. Let R be the region enclosed by the graph of $y = \ln x$, the line $x = 3$, and the x-axis.

a) Find the area of region R by evaluating an antiderivative.

b) Find the volume of the solid generated by revolving region R about the x-axis.

c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated by revolving the region R about the line $x = 3$.

---

3. Consider the differential equation given by $\dfrac{dy}{dx} = \dfrac{-xy}{\ln y}$.

a) Find the general solution of the differential equation.

b) Find the solution that satisfies the condition that $y = e^2$ when $x = 0$. Express your answer in the form $y = f(x)$.

c) Explain why $x = 2$ is not in the domain of the solution you found in part (b).

APPENDIX A

FORMULAS

Formulas from Geometry

Area Formulas

Square

$A = s^2$ where s is the side length
$A = \frac{1}{2}d^2$ where d is the length of the diagonal

Triangle

$A = \frac{1}{2}bh$ where b is the base and h is the altitude
$A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semiperimeter and a, b, and c are the sides
$A = sr$ where s is the semiperimeter and r is the radius of the inscribed circle
$A = \frac{1}{2}ab\sin\theta$ where a and b are two sides and θ is the measure of the angle between a and b

Equilateral Triangle

$A = \frac{1}{4}s^2\sqrt{3}$ where s is the side length
$A = \frac{1}{3}h^2\sqrt{3}$ where h is the altitude

Parallelogram

$A = bh$ where b is the base and h is the altitude

Rhombus

$A = bh$ where b is the base and h is the altitude
$A = \frac{1}{2}d_1d_2$ where d_1 and d_2 are the two diagonals

Kite

$A = \frac{1}{2}d_1d_2$ where d_1 and d_2 are the two diagonals

Trapezoid

$A = \frac{1}{2}(b_1+b_2)h$ where b_1 and b_2 are the parallel bases and h is the distance between them

Cyclic Quadrilateral

$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where s is the semiperimeter and a, b, c, d are the sides

Regular Polygon

$A = \frac{1}{2}ans$ where a is the apothem, n is the number of sides, and s is the side length
$A = \frac{1}{2}ap$ where a is the apothem and p is the perimeter

Ellipse

$A = ab\pi$ where a is half the major axis and b is half the minor axis

Circle

$A = \pi r^2$ where r is the radius
$A = \frac{1}{2}Cr$ where C is the circumference and r is the radius
$A = \frac{1}{4}\pi d^2$ where d is the diameter

Sector of a Circle

$A = \frac{1}{360^\circ}\pi ar^2$ where a is the angle (in degrees) that intercepts the arc and r is the radius
$A = \frac{1}{2}ar^2$ where a is the angle (in radians) that intercepts the arc and r is the radius

Surface Area Formulas

Prism and Cylinder
$S = 2B + ph$ where B is the area of the base, p is the perimeter of the base, and h is the height

Pyramid and Cone
$S = B + \frac{1}{2}ps$ where B is the area of the base, p is the perimeter of the base, and s is the slant height of a lateral face

Sphere
$S = 4\pi r^2$ where r is the radius

Volume Formulas

Prism and Cylinder
$V = Bh$ where B is the area of the base and h is the height

Pyramid and Cone
$V = \frac{1}{3}Bh$ where B is the area of the base and h is the height

Sphere
$V = \frac{4}{3}\pi r^3$ where r is the radius

Greek Alphabet

Upper case	Lower case		Upper case	Lower case	
A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ	epsilon	R	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

Trigonometric Values

$\sin 0 = 0$

$\cos 0 = 1$

$\tan 0 = 0$

$\sin\dfrac{\pi}{6} = \dfrac{1}{2}$

$\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

$\tan\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{3}$

$\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

$\cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

$\tan\dfrac{\pi}{4} = 1$

$\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

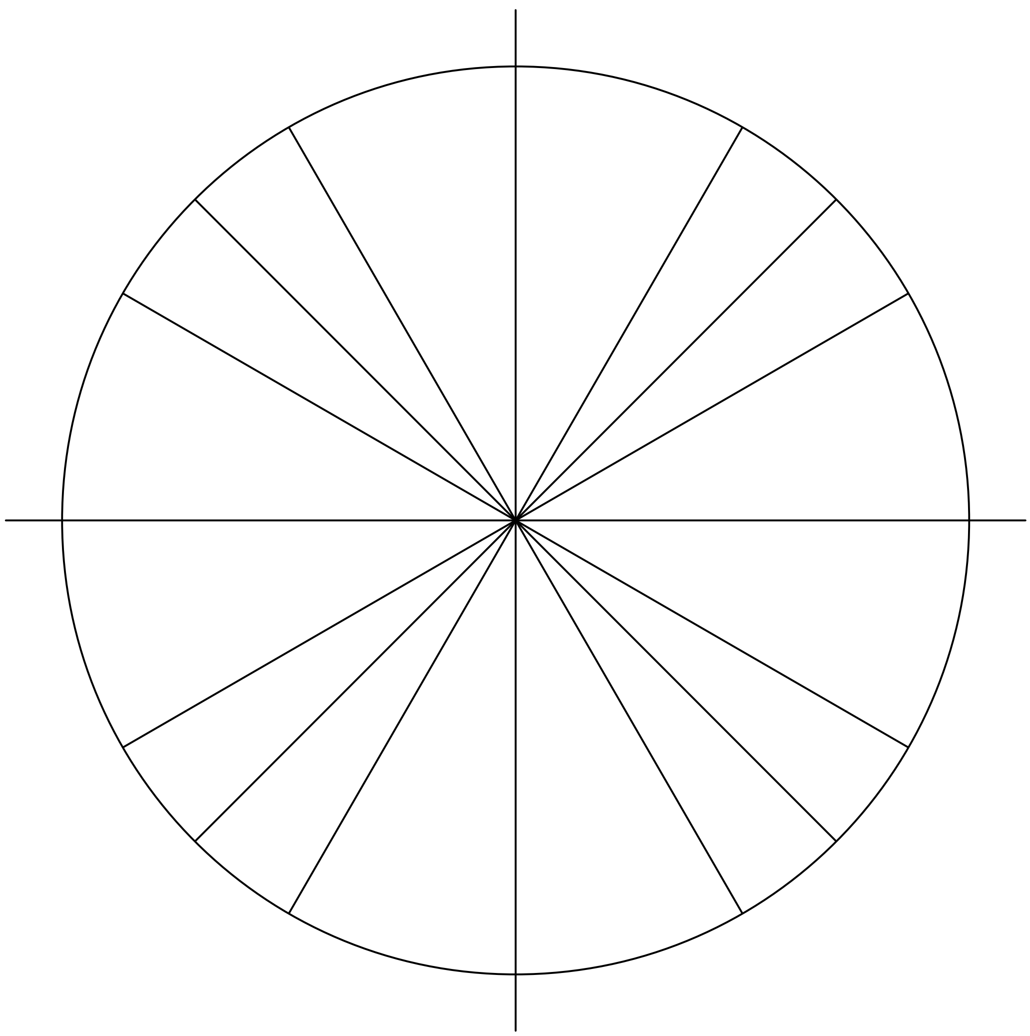
$\cos\dfrac{\pi}{3} = \dfrac{1}{2}$

$\tan\dfrac{\pi}{3} = \sqrt{3}$

$\sin\dfrac{\pi}{2} = 1$

$\cos\dfrac{\pi}{2} = 0$

$\tan\dfrac{\pi}{2}$ is undefined



Useful Trigonometric Identities

Triangle Ratios

$$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\csc x = \frac{1}{\sin x} = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec x = \frac{1}{\cos x} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{\text{adjacent}}{\text{opposite}} \qquad \tan x = \frac{\sin x}{\cos x} = \frac{\text{opposite}}{\text{adjacent}}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad \cot^2 x + 1 = \csc^2 x$$

Double Angle Identities

$$\sin 2x = 2 \sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

Power Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Sum and Difference Identities

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Law of Sines

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

APPENDIX B

SUCCESS IN MATHEMATICS

Math Study Skills

The material is adapted from the Mathematics Department webpage of Saint Louis University, `http://euler.slu.edu/Dept/SuccessinMath.html`.

Active Study vs. Passive Study

Be *actively* involved in managing the learning process, the mathematics, and your study time:

- Take responsibility for studying, recognizing what you do and don't know, and knowing how to get the Instructor to help you with what you don't know.
- Attend class every day and take complete notes. Instructors formulate test questions based on material and examples covered in class as well as on those in the book.
- Be an active participant in the classroom. Read ahead in the textbook; try to work some of the problems before they are covered in class. Anticipate what the Instructor's next step will be.
- Ask questions in class! There are usually other students wanting to know the answers to the same questions you have.
- Go to office hours and ask questions. The Instructor will be pleased to see that you are interested, and you will be actively helping yourself.
- Good study habits throughout the semester make it easier to study for tests.

Studying Math is Different from Studying Other Subjects

- Math is learned by doing problems. It is vital that you *DO THE HOMEWORK.* The problems help you learn the formulas and techniques you do need to know, as well as improve your problem-solving prowess.
- A word of warning: Each class builds on the previous ones, all semester long. You must keep up with the Instructor: attend class, read the text and do homework every day. Falling a day behind puts you at a disadvantage. Falling a week behind puts you in deep trouble.
- A word of encouragement: Each class builds on the previous ones, all semester long. You're always reviewing previous material as you do new material. Many of the ideas hang together. *Identifying and learning the key concepts means you don't have to memorize as much.*

College Math is Different from High School Math

A College math class covers material at about twice the pace that a High School course does. You are expected to absorb new material much more quickly. Tests are probably spaced farther apart and so cover more material than before. The Instructor may not even check your homework.

- Take responsibility for keeping up with the homework. Make sure *you* find out how to do it.
- You probably need to spend more time studying per week – you do more of the learning outside of class than in High School.
- Tests may seem harder just because they cover more material.

Study Time

You may know a rule of thumb about math (and other) classes: at least 2 hours of study time per class hour. But this may not be enough!

- Take as much time as you need to do all the homework and to get complete understanding of the material.

- Form a study group. Meet once or twice a week (also use the phone, email, and instant messaging). Go over problems you've had trouble with. Either someone else in the group will help you, or you will discover you're all stuck on the same problems. Then it's time to get help from your Instructor.

- The more challenging the material, the more time you should spend on it.

Studying for a Math Test

Everyday Study is a Big Part of Test Preparation

Good study habits throughout the semester make it easier to study for tests.

- *Do the homework when it is assigned.* You cannot hope to cram 3 or 4 weeks worth of learning into a couple of days of study.

- On tests you have to solve problems; homework problems are the only way to get practice. As you do homework, make lists of formulas and techniques to use later when you study for tests.

- Ask your Instructor questions as they arise; don't wait until the day or two before a test. The questions you ask right before a test should be to clear up minor details.

Studying for a Test

- Start by going over each section, reviewing your notes and checking that you can still do the homework problems (*actually work the problems again*). Use the worked examples in the text and notes – cover up the solutions and work the problems yourself. Check your work against the solutions given.

- *You're not ready yet!* In the book each problem appears at the end of the section in which you learned how do to that problem; on a test the problems from different sections are all together.
  - Step back and ask yourself what kind of problems you have learned how to solve, what techniques of solution you have learned, and how to tell which techniques go with which problems.
  - Try to explain out loud, in your own words, how each solution strategy is used. If you get confused during a test, you can mentally return to your verbal "capsule instructions." Check your verbal explanations with a friend during a study session (it's more fun than talking to yourself!).
  - Put yourself in a test-like situation: work problems from review sections at the end of chapters, and work old tests if you can find some. It's important to keep working problems the whole time you're studying.

- Also:
  - Start studying early. Several days to a week before the test (longer for the final), begin to allot time in your schedule to reviewing for the test.
  - Get lots of sleep the night before the test. Math tests are easier when you are mentally sharp.

Taking a Math Test

Test-Taking Strategy Matters

Just as it is important to think about how you spend your study time (in addition to actually doing the studying), it is important to think about what strategies you will use when you take a test (in addition to actually doing the problems on the test). Good test-taking strategy can make a big difference to your grade!

Taking a Test

- First look over the entire test. You'll get a sense of its length. Try to identify those problems you definitely know how to do right away, and those you expect to have to think about.

- Do the problems in the order that suits you! Start with the problems that you know for sure you can do. This builds confidence and means you don't miss any sure points just because you run out of time. Then try the problems you think you can figure out; then finally try the ones you are least sure about.

- Time is of the essence – work as quickly and continuously as you can while still writing legibly and showing all your work. If you get stuck on a problem, move on to another one – you can come back later.

- Show all your work: make it as easy as possible for the Instructor to see how much you do know. Try to write a well-reasoned solution. If your answer is incorrect, the Instructor will assign partial credit based on the work you show.

- Never waste time erasing! Just draw a line through the work you want ignored and move on. Not only does erasing waste precious time, but you may discover later that you erased something useful (and/or maybe worth partial credit if you cannot complete the problem). You are (usually) not required to fit your answer in the space provided - you can put your answer on another sheet to avoid needing to erase.

- In a multiple-step problem outline the steps before actually working the problem.

- Don't give up on a several-part problem just because you can't do the first part. Attempt the other part(s) – the actual solution may not depend on the first part!

- Make sure you read the questions carefully, and do all parts of each problem.

- Verify your answers – does each answer make sense given the context of the problem?

- If you finish early, check every problem (that means rework everything from scratch).

Getting Assistance

When

Get help as soon as you need it. Don't wait until a test is near. The new material builds on the previous sections, so anything you don't understand now will make future material difficult to understand.

Use the Resources You Have Available

- Ask questions in class. You get help and stay actively involved in the class.

- Visit the Instructor's Office Hours. Instructors like to see students who want to help themselves.

- Ask friends, members of your study group, or anyone else who can help. The classmate who explains something to you learns just as much as you do, for he/she must think carefully about how to explain the particular concept or solution in a clear way. So don't be reluctant to ask a classmate.

- Find a private tutor if you can't get enough help from other sources.

- All students need help at some point, so be sure to get the help you need.

Asking Questions

Don't be afraid to ask questions. Any question is better than no question at all (at least your Instructor/tutor will know you are confused). But a good question will allow your helper to quickly identify exactly what you don't understand.

- An unhelpful comment: "I don't understand this section." The best you can expect in reply to such a remark is a brief review of the section, and this will likely overlook the particular thing(s) which you don't understand.
- Good comment: "I don't understand why $f(x+h)$ doesn't equal $f(x)+f(h)$." This is a very specific remark that will get a very specific response and hopefully clear up your difficulty.
- Good question: "How can you tell the difference between the equation of a circle and the equation of a line?"
- Okay question: "How do you do #17?"
- Better question: "Can you show me how to set up #17?" (the Instructor can let you try to finish the problem on your own), or "This is how I tried to do #17. What went wrong?" The focus of attention is on your thought process.
- Right after you get help with a problem, work another similar problem by yourself.

You Control the Help You Get

Helpers should be coaches, not crutches. They should encourage you, give you hints as you need them, and sometimes show you how to do problems. But they should not, nor be expected to, actually do the work you need to do. They are there to help you figure out how to learn math for yourself.

- When you go to office hours, your study group or a tutor, have a specific list of questions prepared in advance. You should run the session as much as possible.
- Do not allow yourself to become dependent on a tutor. The tutor cannot take the exams for you. You must take care to be the one in control of tutoring sessions.
- You must recognize that sometimes you do need some coaching to help you through, and it is up to you to seek out that coaching.

APPENDIX C

ANSWERS

22. 1

23. ± 2

24. 2

25. none

26. 2

27. -1 and 5

34. $(\frac{4}{15}, -\frac{1}{5})$

35. $y = -\frac{2}{3}(x-2)+4$ and $y = -\frac{2}{5}(x+3)+6$

36. $y = -3(x-1)+5$ and $y = \frac{1}{3}(x+1)-1$

37. $(-\frac{27}{4}, \frac{5}{8})$

38. $f(x) = 4x-7$

40. 20

42. $\frac{7}{2}$

43. $y = \frac{1}{4}x - 1$

46. $(3u-2)(11u-5)$

47. not factorable

48. $(x-9)(x-3)$

54. $(2x+1)(2x-1)(4x^2+2x+1)(4x^2-2x+1)$

55. $(x+7)(x^2-x+19)$

57. $(p+2)(p^2-5)(p^2-2p+4)$

60. $\frac{1}{4}(7x+7)$

62. $\dfrac{y^2}{y-1}$

63. $\dfrac{xy(x+y-2)}{(x-1)(y-1)}$

64. $\frac{3}{7}(21-\sqrt{7})$

67. $\frac{1}{13}(11-6\sqrt{3})$

69. $3(\sqrt{2x+3}+\sqrt{2x})$

70. $5(\sqrt{x+5}+\sqrt{5})$

71. $\frac{2}{77}(29-13\sqrt{15})$

73. all real numbers

74. all real numbers

75. all real numbers

76. $\{x|x \geq -1\}$

77. all real numbers

78. $\{x|x \neq 0\}$

81. (a) $\{x|x \neq 3\}$ (b) $\frac{2}{3}$ (c) $x=3$, $y=1$

82. (a) $\{x|x \neq -4, 2\}$ (b) 0 (c) $x=-4$, $x=2$, $y=0$

84. (a) $\{x|x \neq 4\}$ (b) $-\frac{3}{4}$ (c) $x=4$

88. $(3z+8)(5z+4)$

91. $(3x+y)(9x^2-3xy+y^2)$

92. $(2w^2+1)(w-5)$

97. $-\frac{1}{2}$

99. -5

101. $\frac{2}{3}$

102. 0

104. $\frac{3}{4}\sqrt{2}$

109. $x=1, y=2$

110. $x=\frac{1}{3}, y=\frac{2}{5}$

111. $k=-2$

114. (a) 5 (b) 5 (c) 5 (d) -5

117. (a) 3 (b) -3 (c) d.n.e. (d) undefined

131. $x=y=\frac{1}{3}$

132. $k=-3, m=3$

134. (a) $\{x|x \neq 0, -2\}$ (b) none (c) $x=-2$

135. (a) $\{x|x \neq 0\}$ (b) none (c) $x=0$

136. -10

137. -1

138. $\frac{5}{7}$

139. 20

140. 4

141. $-\frac{1}{2}$

152. true

153. true

154. false

155. true

156. true

157. true

158. false

159. false

160. false

161. false

162. true

163. false

164. $\frac{1}{2\sqrt{3}}$

165. $-\frac{1}{4}$

166. 1

167. 0

168. e

169. $\frac{1}{e}$

170. f

171. f

172. $-\infty$

173. ∞

174. d.n.e

175. 0

176. $-\infty$

177. ∞

178. d.n.e.

179. d.n.e.

180. k

181. f

182. i

183. 0

184. undefined

185. k

187. x^2+2x+4

188. (a) 10 (b) 30 (c) 20 (d) 40.2

189. $c+3$

190. $\dfrac{\sqrt{c}-1}{c-1}$

191. $2(c-1)$

192. 2

193. $-\frac{3}{c}$

194. $-2(c^2+c+1)$

195. xy

196. $3x$

197. $y+3$

198. $x+2y$

199. $5x+\ln 6$

200. $\frac{x^3}{25}$

201. $\{x|x>0\}$, 0

203. $\{x|x\neq \frac{1}{2}\}$, $\sqrt[3]{-7}$

204. $\left\{x|x<-\sqrt{\frac{1}{2}}, x>\sqrt{\frac{1}{2}}\right\}$, none

206. $\{x|x<3\cup x>5\}$, $\ln 15$

210. (a) 2 (c) $y=2x$ (d) $2x+1+\Delta x$

211. (a) 19 (b) 1

212. (a) $\frac{4}{3}$ (b) 1

213. (a) $-\frac{4}{\pi}$ (b) $-\frac{4\sqrt{3}}{\pi}$

214. (a) $-\frac{2}{\pi}$ (b) 0

215. $-\frac{5}{2}$

217. $-\frac{3}{2}$

218. 0

219. ∞

220. ∞

223. 0

224. d.n.e.

227. d.n.e.

230. d.n.e.

231. -7

233. 2

235. -2

238. 0

239. all reals except 2

240. all real numbers

241. all reals except $[1,2)$

242. all real numbers

243. all reals except 1

244. all reals except ± 1

254. $\frac{1}{5}\ln 7$

255. 1

256. $\frac{1}{3}(\ln 12-7)$

257. e^2-1

258. 0

259. 9

260. 9

261. 1

262. $\pm\sqrt{\frac{\ln 7}{\ln 3}}$

263. 1

267. -3, 0, 3

270. -1, 1, 5

272. 2

273. none

275. 0

276. 0, 2

278. -3

280. -1

281. 0

282. 0, $\frac{3}{5}$

283. 7

284. $\pm\sqrt{\frac{5}{8}}$

285. 0

291. neither

300. none

301. -7

302. 0

303. 3

304. $\frac{1}{2}$

305. 1

306. (b)-(g) yes (h)-(k) no (l) all reals except 0, 1, 2 (m) 0 (n) 2

307. no, there is a hole where $x=-2$

308. no, there is a hole where $x=-4$

309. $a=2$

310. $a=4$

311. $a=-1$, $b=1$

312. $a=4$

313. C

314. B

315. D

316. E

317. D

318. C

319. C

320. E

321. A

322. A

323. C

324. D

325. D

326. B

327. C

328. (a) 2 (b) -2 (c) $-\infty$ (d) $-\infty$ (e) $x=0$ (f) $y=\pm 2$

329. (a) $\{x|x\neq 0\}$ (b) none (c) none (d) 0 (e) 1 (f) d.n.e.

330. (a) odd (b) all nonzero multiples of π (c) d.n.e.

331. (b) all reals in $(0,1)$ and $(1,2)$ (c) 2 (d) 0

347. $(5x^{3/5}-7x^{4/3})(5x^{3/5}+7x^{4/3})$

348. $2x^{-7/3}(2-3x^{2/3}+6x^{4/3})$

349. $x^{-3}(x+1)(x^5-1)$

350. $\frac{2}{3}x^{4/3}(2x^{1/3}+3)(x^{1/3}+4)$

351. $\dfrac{(2x+3)\sqrt{x^2+3x}}{2x(x+3)}$

352. $\sqrt{x+3}$

353. $x^{-2/3}(x^{1/3}+1)(2x+5+2x^{2/3})$

354. $\dfrac{-2(x-4)}{3x^{7/3}(x-2)^{1/3}}$

355. $\dfrac{x^2-7}{2x^{3/2}\sqrt{x^2+7}}$

356. $\dfrac{11-x}{2(x-3)^2\sqrt{x-7}}$

371. 1

372. $2x - 3 + 5x^{-2} - 14x^{-3}$

373. $12x + 13$

374. $\frac{1}{2}x^{-1/2} + \frac{34}{3}x^{-1/3}$

375. $2\pi x^2 + 20\pi x$

380. $14x$, 14, $14x$, 28

381. no

382. no

383. yes

384. no

385. yes

386. yes

392. $(-2, -5)$

394. 6π

396. yes

397. (a) $8(x-2) = y-1$ (b) $[-4, \infty)$ (c) $8(x-2) = y-1$ and $8(x+2) = y-1$

398. no

399. no

400. yes

401. yes

402. no

403. no

404. no

405. no

409. (a) $6t$ (b) -6

410. (b) $-(x+1) = y+2$

412. (a) 280 (b) mg/day

428. a, d, and e

429. d.n.e.

430. 0

431. 0

432. $\frac{\pi}{2}$

433. $-\frac{\pi}{2}$

434. $\frac{\pi}{2}$

435. only one is even, only one is neither

436. $0, \frac{2\pi}{3}, \frac{4\pi}{3}$

437. $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

438. $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

439. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

440. $\frac{2\pi}{3}, \frac{\pi}{2}, \frac{4\pi}{3}, \frac{3\pi}{2}$

441. π

442. (a) 9000 gal (b) 300 gal/hr (c) yes, the tank's volume is zero at $t = 30$ hrs

444. (a) yes (b)-(d) no

445. none must be true

446. (a) $a = b+2$ (c) $a = 3$, $b = 1$

453. $-12y^2(y^3-5)^{-5}$

455. $\dfrac{-3p^4 + 21p^2 - 36p + 10}{(p^3 + 2p - 6)^2}$

456. $\dfrac{-3}{2x^{5/2}\sqrt{5}}$

458. $\dfrac{-z}{(36 - z^2)^{3/2}}$

460. $\dfrac{10u + 5}{6\sqrt{u-1}(2u+3)^{2/3}}$

461. $\dfrac{15}{(x+5)^2}$

463. $\dfrac{-20(x+5)}{(x-5)^3}$

464. $\dfrac{7}{(1-3x)^2}$

466. $\dfrac{-24x^2 + 80x + 47}{(5-3x)^2}$

483. $-3(x - \frac{\pi}{4}) = y - 4$; $\frac{\pi}{2}$

485. $\cos x$

486. $\pi(x-1) = y-2$

487. $\csc\theta(\cot^2\theta + \csc^2\theta)$

488. $\sec\theta(\tan^2\theta + \sec^2\theta)$

489. $2\sin\theta$

490. $-\sin\theta - \cos\theta$

497. $10(x-1) = y-2$

499. no

500. no

501. no

502. yes

503. no

504. no

505. (a) 5 (b) 0 (c) 8 (d) 2 (e) 6 (f) -1

517. $y = 1$

518. (a) $\frac{5}{4}(x-4) = y-2$ and $\frac{4}{5}(x-2) = y-4$ (b) 0 and $3\sqrt[3]{2}$ (c) 0 and $3\sqrt[3]{4}$

519. $(\pm\sqrt{7}, 0)$; slope is -2

520. $(3, -1)$

521. (a) $\dfrac{3x^y - y^2}{2xy - x^3}$ (b) at $(1, -2)$ the tangent is $2(x-1) = y+2$, at $(1, 3)$ the tangent is $y = 3$ (c) $\sqrt[5]{-24}$

530. $AC = \frac{5}{4}\sqrt{29}$, $BC = \frac{25}{2}$

531. $16\sqrt{3}$

532. 50π

533. 15

534. $3\sqrt[3]{9}$

535. $6\sqrt{3}$

536. $2\sqrt{3}$

545. (a) 34994 dollars/week (b) 200 dollars/week (c) 34794 dollars/week

546. $\frac{5}{16}\sqrt{3}$ m/hr

547. 18 m/sec

548. 3 ft/sec

550. $(\frac{1}{4}, \frac{1}{2})$

551. (a) $s = \frac{7}{8}d$ (b) $\frac{35}{8}$ ft/sec

552. 1 ft/min; 40π ft^2/min

553. ≈ 7.1 in/min

554. 12 in^3/sec

555. (a) $\pm\frac{5}{3}$ units/sec (b) ± 24

572. $\ln x$

579. $\dfrac{2^x \ln\frac{2}{5} + \ln 5}{5^x}$

582. $\dfrac{3}{5(3x-2)}$

584. $\dfrac{x\ln x - x + 2}{x(\ln x)^2}$

585. $-6x^2e^{-2x^3}$

586. $\dfrac{e^x(x-3)}{x^4}$

587. $-2x^{-3}$

588. $6(x-1)10^{3x^2-6x}\ln 10$

589. $3^{2x}2^{3x^2}(\ln 9 + 6x\ln 2)$

590. $\dfrac{2xy+y}{3xy-x}$

591. $\frac{1}{3}(4x+12y-17)$

592. $\dfrac{y}{ye^y+1}$

593. $\dfrac{4\cos(x-3y)}{1+12\cos(x-3y)}$

594. $\dfrac{2}{3\cos y-2}$

595. $\dfrac{\sin(x-2y)}{2\sin(x-2y)-3}$

598. $\dfrac{-5\csc^2 5x}{2\sqrt{\cot 5x}}$

599. $24\cos 16x$

601. $-6\sin 6x$

602. $e^{\sin x}\cos x$

603. $-3^{\cos x}\ln 3\sin x$

604. $\frac{2}{\ln 3}\cot 2x$

605. $6x$

606. $e^{3x}(\sec^2 x + 3\tan x)$

607. $\dfrac{-2e^{1/x^2}}{x^3}$

608. $\frac{1}{2}xe^{x^2/4}$

610. $e^{\tan x}(1+x\sec^2 x)$

625. $\frac{2}{\sqrt{3}}$

626. $\frac{1}{2}$

627. $\frac{1}{51}$

628. $\frac{1}{2}$

629. $\frac{\sqrt[3]{25}}{2}$

631. $\dfrac{-1}{\sqrt{-x^2+3x-2}}$

634. $\dfrac{-3}{x^2+9}$

635. $\dfrac{1}{|x|\sqrt{x^2-1}}$

636. $\dfrac{-4}{\sqrt{2-4x^2}}$

637. $\dfrac{-1}{\sqrt{2x-x^2}}$

638. $y = ex$

644. (a) 6.7 million ft^3/acre (b) 0.073 and 0.04 million ft^3/acre per year

647. (b) 50 (c) 25 (d) $1-0.04x$ (e) 0

648. (a) $x'(t) = \dfrac{1}{1+t^2}$ is always positive (b) $x''(t) = \dfrac{-2t}{(1+t^2)^2}$ is always negative (c) $\frac{\pi}{2}$

649. (a) $\frac{10}{\sqrt{2}}$ (b) left -10, right 10 (c) when $t=-10$, $v=0$ and $a=10$, when $t=10$, $v=0$, $a=-10$ (d) at $t=-\frac{\pi}{4}$, $v=-10$, speed $=10$, $a=10$

650. (a) $2x$ (b) $2x$ (c) 2 (d) 2 (e) yes

651. (a) $x=-1$ (b) ≈ -1

652. (a) $\mathbb{R}$ (b) $\dfrac{-\cos x}{2\sqrt{1-\sin x}}$ (c) $\{x|x \neq \frac{\pi}{2}+2\pi n, n\in\mathbb{Z}\}$ (d) $y=-\frac{1}{2}x+1$

653. (a) $-\dfrac{2x+y}{x+2y}$ (c) $(6,-3)$ and $(-6,3)$

654. (a) $\frac{24}{\pi}$ in/sec (b) $\frac{120}{\pi}-30$ in^2/sec

655. (a) $\frac{3}{\sqrt{5}}$ m/sec (b) 150 m^2/sec (c) $\frac{3}{125}$ radian/sec

656. (a) $a=0, b=9, c=4$ (b) $x=\pm 2$ (c) $y=0$

658. E

659. D

660. D

661. D

662. E

663. E

664. E

665. B

666. B

667. C

668. D

669. D

670. B

671. D

672. D

673. A

674. D

675. E

681. (a)-(c) no (d) $(\pm 3, 0)$, $(\pm\sqrt{3}, 6\sqrt{3})$, and $(0,0)$

682. $\{x|0<x<5\}$, extreme values are 0 and 144

693. $c=1$

694. $c=\sqrt{\frac{7}{3}}$

695. $\frac{5}{6}$

696. $-1, 0, 1$

698. $0, \frac{1}{5}$

699. No, Rolle's Theorem does not apply since f is not continuous on $[0,1]$.

709. $a=-\frac{19}{9}, b=\frac{11}{3}$

710. $a=6, b=27, c=36, d=16$

711. (a) $(0,0)$ (b) ccup for $x>0$, ccdown for $x<0$

714. (a) $\left(\pm\frac{2}{\sqrt{3}}, \frac{9}{16}\right)$ (b) ccup for $x<-\frac{2}{\sqrt{3}}$ and $x>\frac{2}{\sqrt{3}}$, ccdown for $-\frac{2}{\sqrt{3}}<x<\frac{2}{\sqrt{3}}$

715. (a) $(3,6)$ (b) ccup for $x>3$, ccdown for $x<3$

717. $a=-3, b=-6$

718. $a=-1, b=-3, c=-5$

722. (b) max at $x=-2$, min at $x=0$

726. (a) $\{x|x \neq \pm 3\}$ (b) 0 (c) 0 (d) min at $(0,0)$ (e) inc for $x < -3$ and $-3 < x < 0$, dec for $0 < x < 3$ and $x > 3$ (f) none (g) ccup for $x < -3$ and $x > 3$, ccdown for $-3 < x < 3$

729. (a) $\{x|x > 0\}$ (b) 1 (c) none (d) max at $(e, \frac{1}{e})$ (e) inc for $0 < x < e$, dec for $x > e$ (f) $\left(e^{3/2}, \frac{3}{2e^{3/2}}\right)$ (g) ccup for $x > e^{3/2}$, ccdown for $0 < x < e^{3/2}$

732. 3.84

733. (a) mins at $x = -2.5$ and $x = 2$, max at $x = 0$ (b) ccup for $-3 < x < -1$ and $1 < x < 3$, ccdown for $-1 < x < 1$ and $3 < x < 4$

735. (a) at $t = \frac{2}{3}$ $x = \frac{53}{81}$, at $t = -1$ $x = \frac{1}{12}$ (b) at $t = \frac{1}{12}$ $x = -\frac{2}{3}\sqrt{3}$

736. (a) 0 (b) 6 (c) always right

737. (a) $v(t) = -2\pi t \sin(\frac{\pi}{2}t^2)$ (b) $a(t) = -2\pi(\sin(\frac{\pi}{2}t^2)$ $+\pi t^2 \cos(\frac{\pi}{2}t^2))$ (c) right for $-1 < t < 0$, left for $0 < t < 1$ (d) 0

738. (a) $3\pi t - 3\pi t \cos(\frac{3\pi}{2}t^2)$ (b) $3\pi - 2\pi \cos(\frac{3\pi}{2}t^2) + 9\pi^2 t^2 \sin(\frac{3\pi}{2}t^2)$ (c) 0, $\sqrt{\frac{4}{3}}$, $\sqrt{\frac{8}{3}}$ (d) 0, 2π, 4π

740. (a) $4e^{3t} - 8$ (b) $12e^{3t}$ (c) $\frac{1}{3}\ln 2$ (d) $\frac{8}{3}(1 - \ln 2)$

741. (a) 135 sec (b) $\frac{5}{73}$ furlongs (c) $\frac{1}{13}$ furl/sec (d) the last and first furlong

746. one piece 14.8 m, other 15.2 m; use all iron to make the triangle

747. $8 \times 8 \times 4$ cm

748. 42

749. 225×150 m

750. \$2.95

751. $\frac{\pi}{4}$

752. (a) ≈ 578.7 cm^3 (b) $616\frac{2}{3}$ cm^2

753. (a) $[0, B]$, max dosage, scale factor (b) $\frac{2}{3}B$ (c) $\frac{4}{27}AB^3$ (d) $\frac{1}{3}B$

754. R^2

755. 10 shipments of 240 players each

756. $\frac{1}{\sqrt{2e}}$

757. (a) $\frac{\pi}{3}, \frac{4\pi}{3}$ (b) 1 (c) $\frac{\pi}{3}, \frac{4\pi}{3}$

759. -1

769. crit pt is $x = 1$, inc for $x < 1$, dec for $x > 1$, extrema at $x = 1$

778. (c) $\arctan x + \dfrac{x}{1 + x^2}$ (e) $-25x^{-2} + 6x^{-1/2}$ (f) $30x^4 - 60x^3 + 20x - 21$ (g) $\dfrac{-2(x^2 + 1)}{(x^2 - 1)^2}$

780. $y = \frac{1}{2}t$

781. (d) $y = e^x$

782. $y' = \cot x$

783. e^3

788. $\frac{\pi}{3}$

789. (a) 4, 0 (b) -1, -1, 1, $\frac{1}{2}$ (c) 0, $-\frac{3}{2}$

790. (a) 1, $\frac{3}{4}$ (b) positive (c) zero

791. (a) $h' = 0$ (b) $k' = 0$

793. $\frac{1}{4}$

794. (a) odd (b) $\dfrac{1 + \cos x + x \sin x}{\cos^2 x}$ (c) $y = 2x$

795. A

796. E

798. (a) max at $x = -1$, mins at $x = \pm 3$ (b) $x = 0$, $x = 1$

799. (b) $x = 0$ (c) everywhere

800. $\mathbb{R}$, min at $(0, \frac{1}{10})$

801. $\mathbb{R}$, max at $(0, 10)$

803. $\{x|x \neq -1\}$, no extrema

804. $\{x|x > 0\}$, no extrema

806. $e^{-x}(x - 2)$

807. $e^x(x^2 + 4x + 2)$

808. $e^{x+e^x}(1 + e^x)$

810. (a) $\dfrac{-2xy}{x^2 + y^2}$ (b) $y = \frac{4}{5}x - \frac{13}{5}$ (c) $\sqrt[3]{-13}$

811. (a) 0, $\frac{\pi}{2}$, π (b) $\frac{\pi}{6} < x < \frac{\pi}{2}$ and $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ (c) min of $-\frac{1}{4}$, max of 2

812. (a) $y = 4x + 2$ and $y = 4x - 2$ (b) 1 (c) 0

813. (a) $x = -2$ (b) $x = 4$ (c) $-1 < x < 1$ and $3 < x < 5$

814. (a) $\{x|x \neq 0\}$ (b) even (c) maxs at $x = \pm 1$ (d) $f(x) \leq \ln \frac{1}{2}$

815. (b) $c \approx 1.579$ (c) $y \approx 1.457x - 1.075$ (d) $y \approx 1.457x - 1.579$

817. (a) $k = -2$, $p = 2$ (b) always inc (c) $(1, 1)$

818. (a) min of $\dfrac{-e^{5\pi/4}}{\sqrt{2}}$, max of $e^{2\pi}$ (b) inc for $0 < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < 2\pi$ (c) π

819. (a) 100 (b) $y = \frac{3}{5}x + 20$ (c) yes, the top 5 ft of the tree

820. C

821. B

822. B

823. A

824. D

825. C

826. D

827. C

828. D

829. E

830. B

831. E

832. A

833. B

834. B

857. $\frac{1}{4}x^4 + 2x + C$

858. $\frac{1}{3}x^3 - x^2 + 3x + C$

859. $\frac{2}{5}x^{5/2} + x^2 + x + C$

860. $\frac{2}{3}x^{3/2} + x^{1/2} + C$

861. $\frac{3}{5}x^{5/3} + C$

862. $-\frac{1}{2}x^{-2} + C$

863. $x - \frac{1}{x} + C$

864. $\frac{2}{7}x^{7/2} + C$

865. $3x + C$

866. $\frac{1}{3}x^3 + \cos x + C$

867. $x + \csc x + C$

868. $\tan\theta + \cos\theta + C$

869. $\sec\theta - \tan\theta + C$

870. $20x^{2/5} + C$

871. $-\frac{9}{2}x^{2/3} + C$

872. $\frac{21}{8}x^8 - \frac{14}{5}x^5 + C$

873. $\frac{7}{4}x - \frac{3}{10}x^{5/2} - \frac{3}{2}x^{1/2} + C$

895. $\frac{8}{375}(5x-2)^{3/2}(15x+4) + C$

896. $-\cos(4x^3) + C$

897. $\sin(4e^x) + C$

898. $3^{3t-1} + C$

899. $\frac{1}{4}(6^{2z^2-3}) + C$

900. $\frac{1}{\ln 32}(2^{5x}) + C$

901. $\frac{2}{5}\sqrt{5x+4} + C$

902. $-\frac{1}{3}(7-3y^2)^{3/2} + C$

903. $\frac{1}{3}\sin(3z+4) + C$

904. $-e^{1/t} + C$

905. $\sec(x + \frac{\pi}{2}) + C$

906. $\frac{2}{3}(\cot\theta)^{3/2} + C$

907. $\frac{1}{2}\ln|x^2+4| + C$

908. $\frac{1}{2}\arcsin(2x) + C$

909. $\arctan(e^x) + C$

919. (a) $v(t) = 18t - t^2 + 19$, $a(t) = 9t^2 - \frac{1}{3}t^3 + 19t - \frac{20}{3}$ (b) 1317 m

920. (a) 87 (b) 87

921. (a) 0.969 mi (b) 22.7 sec, 120 mph

922. (a) 758 gal, 543 gal (b) 2363 gal, 1693 gal (c) 31.4 hrs, 32.4 hrs

923. 799500 ft^3

936. $-\frac{4}{3}\sqrt{2} + 1$

943. -3

947. $\frac{51}{2}$

951. 7

953. π

955. (a) 2 (b) negative (c) $\frac{9}{2}$ (d) 6 (e) 4 and 7 (f) to for $6 < t < 9$, away for $0 < t < 6$ (g) the right side

960. $-\frac{1}{8}(4x^2-1)^{-4} + C$

961. $\frac{1}{2}z^2 - 2z + \frac{5}{z} + C$

962. $\frac{1}{71}(x+7)^{71} + C$

963. $\frac{1}{8}(e^x - 1)^8 + C$

970. $\frac{1}{2}$

972. $\frac{4}{15}$

977. $\frac{4752}{35}$

979. 2

993. $\ln|\sec\theta - 1| + C$

994. $e^{5x} + C$

995. $-\ln(1 + e^{-x}) + C$

996. $-\frac{2}{3}(1 - e^x)^{3/2} + C$

997. $\ln|e^x - e^{-x}| + C$

998. $-\frac{5}{2}e^{-2x} + e^{-x} + C$

999. $\frac{1}{\pi}e^{\sin(\pi x)} + C$

1000. $\ln|\cos(e^{-x})| + C$

1001. $\frac{1}{\ln 3}3^x + C$

1002. $\frac{-1}{\ln 25}5^{-x^2} + C$

1003. $\frac{1}{\ln 9}\ln|1 + 3^{2x}| + C$

1006. $\ln 2$

1009. $4 + 5\ln 5$

1011. $\frac{1}{3}e(e^2 - 1)$

1019. (a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

1020. (a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$

1022. (b) -2

1023. (a) 0 (b) no, yes (c) $\sqrt{12}$ (d) 4π, -4π

1026. 0

1028. $2xf(x^2)$

1037. 16

1039. 86

1040. $\ln 4 + \frac{21}{2}$

1041. $\frac{68}{3}$

1042. $\frac{271}{6}$

1046. (a) $\frac{77}{16}$ (b) $\frac{85}{16}$

1047. (b) 66 degrees

1048. left is 10, right is 7.25

1051. (a) left is 4.06, right is 4.77 (b) 4.36

1058. (a) $\frac{107}{16}$ (b) $\frac{51}{16}$ (c) $\frac{79}{16}$ (d) $\frac{19}{4}$ (e) $\frac{19}{4}$

1061. $\frac{5}{2}$

1062. $\ln\sqrt{2}$

1063. π

1066. $\dfrac{2^{n+1} - 1}{n+1}$

1071. net is $-\frac{9}{2}$, total is $\frac{29}{6}$

1072. net is 0, total is $4e - \frac{4}{e}$

1073. (a) 0 (b) positive (c) -9 (d) $t = 6$ (e) $t = 7$ (f) to from $3 < t < 6$, away from $0 < t < 3$ and $6 < t < 10$ (g) the right side

1074. (a)-(d) true (e) false (f) false (g) true

1082. (a) 9920 (b) $10413\frac{1}{3}$

1084. 4.2 liters

1085. 2.42 gal, 24.83 mpg

1087. $\frac{45\pi}{2}$

1090. (a) 750 ft (b) 550 ft (c) -32, 0

1091. (a) 0 (b) -1 (c) $-\pi$ (d) 1 (e) $y = 2x + 2 - \pi$ (f) -1 and 2 (g) $[-2\pi, 0]$

1092. (a) 63 (b) 234.9

1093. $-\cos x + \sin x + 2$

1094. (a) $f(x) = x^3 + 4x^2 + 3x - 2$ (b) $-\frac{2}{3}$

1095. (a) up (b) -2.049, no (c) 3.827 (d) 1.173

1096. (a) A (b) $\frac{1}{2}A$ (c) 4

1097. (a) 0.316

1098. (a) $x(t) = 4t^3 - 18t^2 + 15t - 1$ (b) $\frac{1}{2}, \frac{5}{2}$ (c) max of 15 (d) 17

1099. (a) -23 (b) 33 (c) 11, 16, -8 (d) a, c

1100. (a) 3 (c) $f(x) = \frac{3}{x}$

1101. (a) 258.6 gal (b) yes (c) 10.785 gal/hr

1102. D

1103. C

1104. D

1105. E

1106. E

1107. B

1108. B

1109. D

1110. A

1111. D

1112. C

1113. B

1114. C

1115. B

1116. D

1117. C

1118. E

1119. E

1125. $\frac{\pi}{8}(\ln 16 - \frac{5}{2})$

1126. $\frac{\pi}{4}(\ln 16 - \frac{5}{2})$

1128. $\frac{512\sqrt{2}}{15}$

1129. (a) $2\sqrt{3}$ (b) 8

1131. $\frac{8}{5}\sqrt{3}$

1133. π

1134. $\frac{\pi}{2}$

1135. 8π

1136. $\dfrac{\pi(e^6 - 1)}{3e^9}$

1137. 4π

1141. $\frac{160625\pi}{14}$

1142. (a) $\frac{459\pi}{10}$ (b) $\frac{405\pi}{7}$ (c) $\frac{243\pi}{20}$

1144. $\frac{\pi}{2}(5e^6 + 1)$

1145. $\pi\left((e-1)\ln 16 - \frac{1}{2e^2}\right)$

1146. 12π

1153. $\frac{1}{27}(31^{3/2} - 8)$

1154. $\frac{2}{3}(10^{3/2} - 5^{3/2})$

1157. $3\sqrt{3}$

1158. $\frac{3\pi}{2}$

1159. $\frac{576\pi}{7}$

1160. $\frac{\pi^2}{2}$

1162. $9\pi\left(\frac{1}{e} - \frac{1}{3}\right)$

1163. $\frac{\pi^2}{3}$

1164. $\frac{2\pi}{15}$

1177. $u = \dfrac{-1}{\sqrt[3]{3(v^2 + C)}}$

1178. $y = \frac{4}{3}x^{3/2} + C$

1179. $y = \frac{1}{6}(3x + 5)^4 + C$

1180. $s = \sin t - \cos t$

1181. $r = \cos(\pi\theta) - 1$

1182. $v = 3\operatorname{arcsec} t - \pi$

1183. $v = 1 + 8\arctan t + \tan t$

1184. $y = x^2 - x^3 + 4x + 1$

1185. $y = \frac{1}{x} + 2x - 2$

1191. $P = \dfrac{-P_0}{P_0 kt - 1}$

1192. $a = \frac{1}{2}, b = -\frac{3}{2}$

1193. A

1194. 16 ft/sec^2

1195. (a) $3000(30)^{t/2}$ (b) 2,700,000 (c) $\approx$ 1.76 hrs

1196. 112500, 2172

1197. \$81000, 2018

1198. (a) $y = 1000000(2^{-t/6})$ (b) 69300 gal/yr (c) $\approx$ 26 yrs

1199. (a) $T(t) = T_a + (T_0 - T_a)e^{-kt}$ (b) T_a (c) 5:12 PM

1204. (a) $\dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

1205. $h(t) = \frac{3}{4}t^2 + 6t + 12$, 75 cm

1207. (a) 500 (b) 1000

1209. $y(t) = c + (y_0 - c)e^{-ktA/V}$

1210. 3.458

1211. 9.822

1212. -0.227

1213. -20.954

1214. (a) -4 (b) $\frac{1024\pi}{105}$ (c) $-\frac{25}{4}$

1215. (a) II (b) III (c) I (d) IV

1216. (a) 2 (b) $\frac{25}{4}$ (c) $\ln\frac{9}{4}$

1217. 2.109

1218. (a) $(-0.767, 0.588)$, $(2, 4)$, $(4, 16)$

1219. (a) 6.234 (b) 6.236 (c) 6.238

1220. $y = (\ln x)^4$

1221. (a) 1.168

1222. (a) $P(t) = 800 - 300e^{-kt}$ (b) $\frac{1}{2}\ln 3$ (c) 800

1223. (a) $(-3, 1)$ (b) $\frac{4}{3}$ (c) $\frac{8\pi}{3}$

1224. (a) $\frac{44\pi}{3}$
(b) $2\pi \displaystyle\int_3^5 x\sqrt{(x+1)^2 - 9}\,dx$

1225. (a) $2\sqrt{4x^2 + 2x}$
(b) $\{x|x > 0\}$ (c) 0 (d) 7

1226. (a) $\frac{20}{9}$
(b) $\pi \int_0^2 (6x+4-4x^2)dx$

1227. (a) $\frac{\pi}{2}(e^4+e^{-4}-2)$
(b) $2\pi(e^2+3e^{-2})$

1228. A

1229. E

1230. C

1231. C

1232. A

1233. C

1234. D

1235. D

1236. C

1237. E

1238. A

1239. E

1240. D

1241. E

1242. C

1253. $y = \left(\frac{x^2}{4} - \frac{x}{8} + \frac{1}{32}\right) e^{4x} + C$

1254. $y = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

1255. $y = -2\sqrt{\theta}\cos\sqrt{\theta} + 2\sin\sqrt{\theta} + C$

1256. $y = \theta\sec\theta - \ln|\sec\theta + \tan\theta| + C$

1257. e^2

1258. $\ln 27 - 3$

1259. $3-e$

1260. $\ln\frac{1}{4} + 5e^2 - 5e - 14$

1262. $\dfrac{e^{2\pi}-1}{2\pi e^{2\pi}}$

1263. (b) -3π (c) 5π

1271. $-\frac{2}{25}\left(\ln\left|\frac{x}{x-5}\right| - \frac{5}{x}\right) + C$

1272. $\frac{1}{125}\left(\ln\left|\frac{x}{x-5}\right| - \frac{5}{x} - \frac{5}{x-5}\right) + C$

1274. $y = \dfrac{2}{2-e^{e^x-1}}$

1275. $y = \dfrac{1}{\cos\theta+1} - 1$

1276. $y = \ln|x-2| - \ln|x-1| + \ln 2$

1277. $|y+1| = \left|\dfrac{6t}{t+2}\right|$

1280. $p(t) = \dfrac{1000e^{4t}}{499+e^{4t}}$, 1.55 days

1287. $y = \sqrt{x^2-4} - 2\operatorname{arcsec}\left(\frac{x}{2}\right)$

1288. $y = \ln\left|\dfrac{x+\sqrt{x^2-9}}{3}\right|$

1289. $y = \frac{3}{2}\arctan(\frac{x}{2}) - \frac{3\pi}{8}$

1290. $y = \dfrac{x}{\sqrt{x^2+1}} + 1$

1291. $\frac{3\pi}{4}$

1292. $\pi(\frac{\pi}{2}+1)$

1294. (a) $(e-2)\pi$ (c) $2\pi(\ln 2 - 1)^2$

1296. $\frac{4}{3}$

1297. (a) 0.968 (b) 7.199

1298. (a) $7B$ (b) $\frac{3}{4}$

1309. $\frac{3}{2}$

1311. 1

1312. d.n.e.

1314. 1

1315. $\sqrt{e}$

1316. 1

1317. 1

1318. e

1319. e

1320. 1

1321. 1

1322. 0

1323. $\ln 2$

1324. 1

1325. -1

1326. 1

1327. $\frac{27}{10}$

1344. -1

1347. divergent

1348. 1

1349. $-\frac{1}{4}$

1351. $\ln 4$

1352. π

1353. $2\pi^2$

1354. $\ln 2$

1359. 0.5 ml

1360. A

1361. (b) negative (c) $-\frac{1}{2}\ln 3$

1362. B

1364. only one is divergent

1365. only one is false

1366. (b) $-3.75(x-1.4) = y - 2.6$ (d) 2.2

1367. (a) 1 (b) 2, 6, 24

1372. (a) $-\frac{13}{6}$ (b) use the Intermediate Value Theorem (c) $y = \frac{21}{2} - 3x$ (d) $\frac{21}{6}$

1373. (a) 1 (b) $\frac{\pi}{2}$ (c) π

1374. $\frac{3}{4}$

1375. 0.726

1376. (a) $x(t) = a - \dfrac{a}{akt+1}$
(b) $x = \dfrac{a(1-e^{(a-b)kt})}{b-ae^{(a-b)kt}}$

1377. $a = \frac{1}{2}$, converges to $-\frac{1}{4}\ln 2$

1378. (a) π (b) $\pi(2e-5)$ (c) $\frac{\pi}{2}(e^2-3)$ (d) $\frac{\pi}{2}(e^2-4e+5)$

1379. (b) $\frac{8\pi}{27}(9(\ln 2)^2 - 6\ln 2 + 2)$

1380. (a) $x = \dfrac{n(e^{(n+1)kt}-1)}{n+e^{(n+1)kt}}$

1381. (a) π

1382. 100π

1384. (a) 1 (b) $2\pi(1-e^{-k}-ke^{-k})$ (c) $\frac{1}{2}(1-e^{-2k})$

1386. C

1387. C

1388. B

1389. D

1390. C

1391. E

1392. E

1393. E

1394. A

1395. A

1396. D

1397. B

1398. A

1399. C

1400. A

1401. C

1402. D

1403. D

1410. 0

1411. 0

1412. 1

1413. 1

1414. 0

1415. divergent

1416. $e^{2/3}$

1417. 1

1418. $\frac{\pi}{2}$

1419. 1

1420. unbounded, divergent

1421. bounded, convergent

1426. $\frac{1}{3}$

1427. divergent

1428. divergent

1429. divergent

1430. 0

1431. e

1432. 0

1433. divergent

1434. divergent

1435. 0

1436. 5

1437. 0

1438. increasing

1439. increasing

1440. decreasing

1441. increasing

1443. oscillating

1447. 1

1448. three are false

1458. $-1 < x < 1$

1459. $-1 < x < 1$

1463. $x < -1$ and $x > 1$

1464. $1 < x < 5$

1465. $\frac{1}{e} < x < e$

1466. $\frac{23}{99}$

1469. $\frac{140}{99}$

1470. $\frac{41333}{33300}$

1471. $\frac{22}{7}$

1472. 28

1473. 8

1477. divergent

1478. divergent

1479. divergent

1485. divergent

1486. divergent

1487. divergent

1488. convergent

1489. divergent

1490. convergent

1491. divergent

1492. convergent

1493. convergent

1494. $a = 1$

1501. $\frac{5}{6}$

1502. converges absolutely

1503. converges absolutely

1504. diverges

1505. converges conditionally

1506. converges condintionally

1507. converges absolutely

1508. converges absolutely

1509. diverges

1510. converges conditionally

1511. diverges

1513. 0.2

1514. 0.00001

1529. $-1 \leq x \leq 1$

1530. $-1 \leq x \leq 1$

1531. $-\frac{1}{4} \leq x < \frac{1}{4}$

1532. $2 < x \leq 4$

1533. $-2 < x \leq 8$

1534. $-\frac{1}{e} \leq x < \frac{1}{e}$

1535. $x = 0$

1537. $-1 < x < 3$, $\frac{4}{4-(x-1)^2}$

1538. $\frac{1}{e} < x < e$, $\dfrac{1}{1 - \ln x}$

1546. (b) $\sum (-1)^{n+1}(\frac{1}{3})^n(x-3)^{n-1}$ (c) $0 < x < 6$

1547. (b) $\sum \dfrac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n-1}}{(2n-1)!}$ (c) $\mathbb{R}$

1550. (b) $\sum \dfrac{(-1)^n \pi^{2n}(x-\frac{1}{2})^{2n}}{(2n)!}$ (c) $\mathbb{R}$

1552. (b) $\sum \dfrac{e^2(x-2)^n}{n!}$ (c) $\mathbb{R}$

1555. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + - \cdots$

1557. divergent

1558. $9 + 5x - 2x^2 + 6x^3$

1559. $-1 + \frac{1}{2}(3x - \pi)^2$

1560. $1 - \frac{1}{8}(4x - \pi)^2$

1561. (a) $-1 < x < 1$ (b) $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - + \cdots$

1572. first and last; second and third

1573. $\mathbf{a} = \mathbf{0}$

1574. $\mathbf{G}(0) = \mathbf{i}$, $\mathbf{G}(\frac{\pi}{3}) = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$

1575. $||\mathbf{F}|| = |t|\sqrt{t^2+4}$, $||\mathbf{G}|| = 1$

1578. yes, for $t = 0$

1585. (a) $\langle 2t, 9t^2\rangle$ (b) $\langle 2, 18t\rangle$ (c) $\langle 2, 9\rangle$ (d) $\sqrt{85}$

1588. (a) $\langle \pi\cos\pi t, \pi\sin\pi t\rangle$ (b) $\langle -\pi^2\sin\pi t, \pi^2\cos\pi t\rangle$ (c) $\langle -\pi, 0\rangle$ (d) π

1591. $-3\mathbf{i} + (4\sqrt{2} - 2)\mathbf{j}$

1593. $\mathbf{R}(t) = [(t+1)^{3/2} - 1]\mathbf{i} - (e^{-t} - 1)\mathbf{j}$

1594. $\mathbf{R}(t) = (8t+100)\mathbf{i} + (8t - 16t^2)\mathbf{j}$

1595. $3\sqrt{13}$

1596. 3π

1597. $\frac{1}{27}(85^{3/2} - 72^{3/2})$

1598. $4\sqrt{13}$

1599. (a) $(-3\sin t)\mathbf{i} + (2\sin t)\mathbf{j}$ (b) $(-3\cos t)\mathbf{i} - (2\sin t)\mathbf{j}$ (c) $-\frac{2}{3}$ (d) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (e) $-\frac{2}{3}(x - \frac{3}{2}\sqrt{2}) = y - \sqrt{2}$

1600. (a) $\mathbf{R}'(t) = \mathbf{i} + 18t\mathbf{j}$, $\mathbf{R}''(t) = 18\mathbf{j}$ (c) $y = 18x - 13$

1601. (a) $\dfrac{-1}{3(1 + \frac{1}{2}\sqrt{3})}$ (b) $(2\sqrt{3} - 4)(x - \pi) = y - 2\sqrt{3}$

1605. (a) $\left\langle 6te^{3t^2}, \dfrac{2 + 8t^2}{t + 2t^3}\right\rangle$ (b) $\langle 6e^2, \frac{10}{3}\rangle$ (c) no (d) $\langle e^2, \ln 3\rangle$

1606. (a) $(\cos t)\mathbf{i} - (2\sin 2t)\mathbf{j}$ (b) $\frac{\pi}{2}, \frac{3\pi}{2}$, $y = 1 - 2x^2, -1 \le x \le 1$

1608. (a) 160 sec (b) 225 m (c) $\frac{15}{4}$ m/sec (d) 80 sec

1609. (a) $t = 2$

1628. $\frac{\pi}{3}, \frac{5\pi}{3}$

1629. $\frac{\pi}{2}, \frac{3\pi}{2}$

1631. $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

1635. $r = 4\cos\theta\csc^2\theta$

1636. $r = \sec\theta$

1637. $r = e^\theta$

1641. (a) $y = -\frac{1}{4}$, $y = 2$, $x = 0$, $x = \pm\frac{3\sqrt{3}}{4}$ (b) $y = \pm 0.267$, $y = \pm 4.343$, $x = -1$, $x = \frac{9}{16}$, $x = -7$

1642. 18π

1643. $\frac{\pi}{8}$

1644. 1

1645. $\frac{\pi}{2}$

1646. $5\pi - 8$

1647. $\pi + 1 - \sqrt{3}$

1648. $12\pi - 9\sqrt{3}$

1649. 8

1650. $\sqrt{2} + \ln(1 + \sqrt{2})$

1651. $2\pi\sqrt{2}$

1652. 8π

1653. (b) $y = 10 - 10x$ (c) 9.236

1654. (b) $\dfrac{2}{e^{2t} - 1}$, $y = \dfrac{2x}{e+1} + 2\ln(e+1) - 2$ (d) $y = 2\ln x$

1655. (a) $1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^n}{(n+1)!}$

1656. (a) $\frac{9\pi}{2}$ (c) $\frac{\pi}{4}$

1657. $-\frac{1}{2} \le x < \frac{1}{2}$

1658. (b) divergent

1659. (a) $4 - 4t^2 + 4t^4 - 4t^6$, $(-1)^n(4t^{2n})$ (b) $4x - \frac{4}{3}x^3 + \frac{4}{5}x^5 - \frac{4}{7}x^7$, $(-1)^n(\frac{4x^{2n+1}}{2n+1})$ (c) $-1 < t < 1$ (d) ± 1

1660. (a) $\sqrt{2t+1} - 5$ (b) $\dfrac{3(\sqrt{2t+1} - 5)^2 - 3}{\sqrt{2t+1}}$ (c) $(-2, -2)$, 3.018

1661. (b) $8 - \pi$

1663. (a) 3.69 (b) 2.645 (c) -1.52

1664. (a) $-3 < x < 3$ (b) $\frac{2}{9}$ (c) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n+1}}$ (d) $\frac{1}{2}$

1665. (a) $\langle 1, 2\rangle$, $\langle \frac{3}{2}, \frac{3}{2}\rangle$ (c) $t = 4$

1666. (a) $(0, 5)$ (c) 3 times (d) $\langle -3\pi\sin\pi t, 5\pi\cos\pi t\rangle$ (e) 5.392

1667. C

1668. E

1669. C

1670. C

1671. A

1672. C

1673. D

1674. C

1675. D

1676. B

1677. E

1678. B

1679. C

1706. surface area is $\frac{\pi}{27}$, need 0.25 cubic inches of glass

1715. $y = \frac{3}{2} - \frac{1}{2}e^{-2x}$

1716. $y = \frac{1}{x}\cot x + \frac{\pi}{2x}$

1717. $y = 1 - 7e^{-x^2/2}$

1719. (a) 10 lbs/min (b) $100 + t$ gal (c) $\frac{4y}{100+t}$ lbs/min (d) $y = 100 + t - \dfrac{150}{(1 + \frac{t}{100})^4}$ (e) ≈ 1.5 lbs/gal

1733. $x_0 > 0 \to \sqrt{2}$, $x_0 < 0 \to -\sqrt{2}$

1736. $y = \pm\sqrt{9 - x^2}$

1737. $y = \pm\sqrt{\frac{4}{9}x^2 - 4}$

1738. $y = -1 \pm \sqrt{4 - \frac{1}{2}(x-1)^2}$

1739. $y = e^{x^2+5}$

1740. $y = \ln|x + 7|$

1741. $y = \dfrac{x}{1 - x}$

1742. 0, 2

1743. -5

1744. 0, -2

1745. 2, 6

1746. $|x^2 - 1|$, $x^2 - 1$

1747. x, x

1748. $(x-1)^2 + 1$, x^2

1749. $x + 1$, $\sqrt{x^2 + 1}$

1750. 1

1751. 0

1752. 1

1753. 0

1754. -1

1755. 1

1756. 2

1757. -2

1758. $e^x(2x^2+4x+3)$

1759. $5^x 2^{x-1}\ln 10$

1760. $\frac{16}{x+1}+\frac{8(4x+1)}{x(2x+1)}-\frac{x}{x^2+4}$

1761. $\frac{14e^{2x}}{2^{2x}+6}+\frac{1}{2(x+4)}+\frac{5(e^{-x}-e^x)}{e^{-x}+e^x}$

1762. $\dfrac{7}{(2-3x)(x+4)}$

1763. $\dfrac{1-\ln x}{3x^2\ln 10}$

1764. $\sec(3x)(4+(4x-1)\tan(3x))$

1765. $\dfrac{3\sec^2 3x\tan 2x-2\tan 3x\sec^2 2x}{\tan^2 2x}$

1766. $\dfrac{(\ln 10)(\cos 2x-2x\sin 2x\ln x)}{x(\ln x)^2}$

1767. $1+\cos^2 x-x\sin 2x$

1768. $2x(\ln 3)(3^{x^2})$

1769. $\dfrac{-x^2-2x+1}{(\ln 10)(x+1)(x^2+1)}$

1770. $2e^{-x}(1-x)$

1771. $\dfrac{6xy^2+3y^2-2xy\ln y}{2x^2-3xy\ln x-6x^2y}$

1772. $\dfrac{2x}{3x^2e^{3y}-12ye^{3y}+4}$

1773. $\dfrac{1-xe^{5y}+2x^2y}{x^2(5e^{5y}-x)}$

1774. (a) $2<x<6, 8<x<10$ (b) $0<x<2, 6<x<8$ (c) 2, 8 (d) 6 (e) $0<x<3, 7<x<9$ (f) $3<x<7, 9<x<10$ (g) 3, 7, 9 (h) -6, 0

1775. (a) 30 (b) $-\frac{9}{4}$ (c) 24 (d) 4 (e) -12

1776. $-\frac{5}{3}$

1777. -1, -3

1778. 12, -4

1779. (b) $-5<x<-4.715$ and $-1.496<x<0.769$ (c) $-5<x<-3.127$ and $-0.26<x<2$ (d) yes, at $x=0.769$ and $x=-4.715$ (e) $x=-3.127$ and $x=-0.26$

1780. (a) $v(t)=4-6t-3t^2$ (b) $a(t)=-6-6t$

1781. $\sqrt{2-\sqrt{2}}$, collide at $t=\frac{7\pi}{8}$

1782. (a) 4% (b) 8% (c) 12%

1783. (a) $\mathbb{R}$, neither (b) $a'(x)$ is -2 for $x<1$, 0 for $1<x<3$, and 2 for $x>3$ (c) min is 2

1784. 120

1785. (a) 2000, -187 (b) 6

1786. (b) $y'=\dfrac{-2x-y}{x+2y}$, $y'=\dfrac{-2x+y}{2y-x}$ (c) $-\frac{1}{2}$ and -2, $\frac{1}{2}$ and 2 (d) let $z=\sqrt{\frac{1}{3}}$: $(z,-2z)$, $(-z,2z)$, $(z,2z)$, $(-z,-2z)$ (f) $y=x$, $y=-x$

1790. (a) $\dfrac{1}{1+x^2}$ (b) $\frac{\pi}{2}$, $-\frac{\pi}{2}$, 0, 0 (c) odd (d) $f'(x)>0$ for all x (e) ccup for $x>0$, ccdown for $x<0$, infl pt at the origin

1792. $\frac{2}{3}\ln(3y^2+2)+C$

1793. $2z^{3/2}-\frac{8}{5}z^{5/2}+2z^{1/2}+C$

1794. $\dfrac{3^{5y}}{5\ln 3}+C$

1795. $-\frac{1}{7}\cos 7x+C$

1796. $\frac{2}{15}(3x^2-5)^{5/4}+C$

1797. $\frac{1}{2}x-\frac{1}{16}\sin 8x+C$

1798. $\frac{1}{2}e^{\sin 2x}+C$

1799. $-\frac{1}{12}\cos^3 4x+C$

1800. $-\sin x+2\cos x+C$

1801. $-\frac{3}{2}x^{-2}-\ln|x|-\frac{1}{3}x^{-3}+C$

1802. $\ln|x|+C$

1803. $\frac{1}{2}\ln|2x-3|+C$

1804. $\frac{1}{4}\sin(4x-5)+C$

1805. $\frac{1}{90}(3x^2-2)^5+C$

1806. $\dfrac{2^{3y^2}}{6\ln 2}+C$

1807. $\ln|\sin x-3|+C$

1808. $\frac{1}{2}\ln|\sec 2x|+C$

1809. $-\frac{1}{5}e^{1/x}+C$

1810. $\frac{1}{2}\ln|e^{2x}-7|+C$

1811. $\frac{1}{9}x^3-\frac{2}{25}x^5-\frac{3}{7}x+C$

1812. $\frac{1}{12}(e^{2x}+3)^6+C$

1813. $\frac{6}{11}x^{11/6}-\frac{10}{19}x^{19/10}+C$

1814. $\frac{1}{4}x^4+C$

1815. $\frac{1}{5}e^{5\sin x}+C$

1816. $\frac{15}{11}x^{11/5}-\frac{5}{3}x^{6/5}+C$

1817. $\frac{1}{4}(x^2+3x-2)+C$

1818. $\frac{1}{3}\ln|x^3-2|+C$

1819. $\frac{1}{6}(x^4-2)^{3/2}+C$

1820. $\frac{2}{5}x^{5/2}+2x^{3/2}+C$

1821. $\frac{1}{4}y^{4/3}-\frac{9}{2}y^{2/3}+C$

1822. $\frac{4}{3}x^{3/2}+6\sqrt{x}+C$

1823. $\frac{3}{2}(x^2+1)^{5/3}+C$

1824. $\frac{1}{525}(5x-4)^6(15x+2)+C$

1825. $\frac{1}{2}\sec 2u+C$

1826. $\frac{1}{2}u+\frac{1}{28}\sin 14u+C$

1827. $\frac{1}{3}\ln|\sin 3x|+C$

1828. $\frac{1}{18}(e^{3x}-5)^6+C$

1829. $\frac{1}{8}(3x^2-1)^{4/3}+C$

1830. $ex+C$

1831. $u^3-3u^{2/3}+C$

1832. $-\frac{1}{3}e^{\cos 3x}+C$

1833. $2\ln|x+3|+C$

1834. $\dfrac{3^{2a}}{\ln 9}+C$

1835. $\sin 5x+C$

1836. $\frac{1}{4}\cos 4x+C$

1837. $x^2+3x-2\ln|x|+C$

1838. $\ln|5+\tan x|+C$

1839. $\frac{3}{8}(3x^2-2)^{4/3}+C$

1840. $\frac{1}{96}(8z+16)^{12}+C$

1841. $\frac{2}{3}(x+2)^{3/2}+C$

1842. $-\frac{1}{6}\cos 6y+C$

1843. $\frac{1}{2}\sin 2x+C$

1844. $\frac{1}{2}\ln|\sec 2x+\tan 2x|+C$

1845. $\frac{1}{24}\sin^6 4x+C$

1846. $\ln|\sin a|+C$

1847. $\sin 2x+C$

1848. $\frac{4}{5}(x-3)^{3/2}(x+2)+C$

1850. $\dfrac{-1}{6(3y^2+2)^4}+C$

1851. $\frac{1}{10}e^{5x^2}+C$

1852. $\frac{1}{5}\sin 5y+C$

1853. $\frac{1}{2}x+\frac{1}{20}\sin 10x+C$

1854. $\frac{1}{15}\sin^3 5x+C$

1855. $\frac{1}{5}\ln|5x^2-3|+C$

1856. $\frac{1}{3}\ln|x|+C$

1857. $-\frac{1}{5}\cos(5\theta-3\pi)+C$

1858. $\dfrac{4^y\ln 5}{\ln 4}+C$

1859. $\frac{1}{6}\ln|\sin(3x^2)|+C$

1860. $x+\frac{7}{2}e^{-2x}+C$

1861. $-\frac{1}{\ln 3}\cos(3^x)+C$

1862. $\sin x-\frac{1}{5}\sin^5 x+C$

1863. $\ln|x-3|+C$

1864. $-\frac{5}{3}\ln|2-3x|+C$

1865. $-\frac{2}{3}\ln|2-3y^2|+C$

1866. $\frac{5}{6}\ln|1+2z^3|+C$

1867. $\frac{1}{2}\ln|3z^2-4z|+C$

1868. $-\frac{2}{3}\ln|1+\cos 3\theta|+C$

1869. $-\frac{1}{2}\ln|3-\sin 4\phi|+C$

1870. $\frac{1}{5}(\ln x)^5+C$

1871. $\frac{1}{4}(\ln x)^4+C$

1872. $\frac{1}{7}\sin^7 x+C$

1873. $-\frac{1}{9}\cos^3 3x+C$

1874. $\frac{1}{2}x-\frac{1}{16}\sin 8x+C$

1875. $-\frac{1}{5}\cos 5x+\frac{1}{15}\cos^3 5x+C$

1876. $-\frac{1}{24}\cos^4(2x)(2\sin^2(2x)+1)+C$

1877. $\frac{1}{2}\tan 2x-x+C$

1878. $\frac{1}{2}\tan(e^{2x})-\frac{1}{2}e^{2x}+C$

1879. $\frac{1}{2}\ln|x|+C$

1880. 9

1881. $\dfrac{14}{\ln 2}$

1882. $2\sqrt{\sqrt{3}+4}-2\sqrt{\sqrt{2}+4}$

1883. $\frac{671}{3240000}$

1884. $10e(e-1)$

1885. $\frac{20}{3}$

1886. $\frac{2}{5}(e^5-1)$

1887. $4-4e^{-4/5}$

1888. $\frac{28}{9}(5\sqrt{5}-2\sqrt{2})$

1889. $40(e^{1/20}-1)$

1890. 0

1891. net and total are $\frac{1}{3}(e^6-1)$

1892. net and total are $e^{12}-1$

1893. net is 0, total is 2

1894. net is $\frac{4}{3}$, total is 2

1895. $\frac{16}{3}$

1896. $\dfrac{19}{\ln(\frac{3}{2}}$

1897. 0

1898. $\frac{\ln 4}{\pi}$

1899. (a) 7 (b) 13 (c) 10 (d) 10

1900. (a) $\frac{88}{25}$ (b) $\frac{48}{25}$ (c) 3 (d) $\frac{8}{3}$

1901. (a) 102 (b) 57 (c) $\frac{651}{8}$ (d) 78

1902. (a)-(d) 56

1903. $\frac{500}{3}$

1904. $\frac{2}{3}$

1905. 2

1906. $\frac{1}{2}$

1907. 0

1908. $\frac{125}{6}$

1909. 12

1910. $\frac{19}{3}+8\ln\frac{2}{3}$

1911. $\frac{\pi}{4}-\ln\sqrt{2}$

1912. $\frac{80}{3}$

1913. 32

1914. $\frac{1}{10}(143-36\sqrt{3})$

1915. $\frac{1}{6}$

1916. $\frac{1}{4}$

1917. $\frac{1}{12}$

1918. $2-\sqrt{2}$

1919. $\frac{8}{3}$

1920. $\frac{5}{6}\sqrt{5}$

1921. $\frac{1}{6}(1000-61\sqrt{61})$

1922. $\frac{8}{3}$

1923. $2-\frac{2}{3}\sqrt{3}$

1924. $-2\sqrt{10x-3}$

1925. $\dfrac{\ln(3\pi x)}{x}$

1926. -2, 5

1927. $\arcsin x+C$

1928. $\ln|\ln x|+C$

1929. $\frac{2}{3}(e^{12}-1)$

1930. $\frac{1}{2}(e^2-21+7\ln 7)$

1931. $\frac{27}{4}$

1932. $\dfrac{(\ln 2+1)^2-1}{2e}$

1933. $-\frac{125}{4}$

1934. $2\sqrt{2}$

1935. (a) $\frac{5}{3}$ (b) $\frac{17}{12}$ (c) $\ln 4$ (d) 2π (e) 4π (f) $4+\int_1^2\sqrt{\frac{4}{x^4}+1}dx$ (g) 5.441

1936. $\frac{\pi}{4}(4-\pi)$

1940. (e) 5.105 (i) between 16 and 17 yrs (j) between 14.2 and 14.3 yrs

1941. $\frac{6}{5}x^{5/2}-4\sqrt{x}+\frac{10}{3}x^{3/2}+C$

1942. $\frac{1}{2}z^2+C$

1943. $\frac{2}{55}(5x^2-3)^{11}+C$

1944. $\dfrac{5^{3x}\ln 7}{3\ln 5}+C$

1945. $\frac{5}{8}e^{4t^2}+C$

1946. $\frac{1}{60}(4x-1)^{3/2}(6x+1)+C$

1947. $3\ln|z|+C$

1948. $-\frac{3}{2}\cos 2z+C$

1949. $\frac{4}{3}\sin 3z+C$

1950. $-2\sqrt{1-w^2}+C$

1951. $\frac{3}{2}x^4-\frac{7}{2}x^{-2}+\frac{1}{3}x^3+C$

1952. $\frac{1}{2}\ln|x^2-1|+C$

1953. $-e^{-x}+C$

1954. $-e^{1/x}+C$

1955. $\ln(e^x+1)+C$

1956. $\frac{1}{3}\sin^3 x+C$

1957. $\frac{1}{2}\ln|\sin(x^2)|+C$

1958. $\frac{4}{9}(x^3+2)^{3/4}+C$

1959. $\frac{3}{4}(x^2+6x)^{2/3}+C$

1960. $x+\dfrac{1}{x+1}+C$

1961. $\frac{1}{3}(2y-3y^2)^{3/2}$

1962. $\frac{2}{3}\ln|1+3u|+C$

1963. $\frac{1}{2}\arctan 2x+C$

1964. $\dfrac{-1}{8(1+4x^2)}+C$

1965. $2\ln|x+1|+\frac{1}{2}x^2+\dfrac{1}{x+1}-2x+C$

1966. $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$

1967. $\frac{1}{3}\tan 3u+C$

1968. $-\ln|\cos\theta|+C$

1969. $\frac{1}{2}\ln|1-\cos 2t|+C$

1970. $\frac{1}{2}\ln|x(x-2)|+C$

1971. $-e^{-x}(x+1)+C$

1972. $\frac{1}{2}(\ln v)^2+C$

1973. $\frac{1}{3}u^3+C$

1974. $\frac{4}{3}$

1975. $\frac{10}{9}$

1976. 2

1977. $\dfrac{2(e^{5/2}-1)}{e^{3/2}}$

1978. 0

1979. 2

1980. 6

1981. $-\frac{116}{15}$

1982. $\frac{1}{2}(e^3-1)$

1983. $1-\frac{\pi}{4}$

1984. $2-\sqrt{3}$

1985. 1

1986. $\frac{3}{2}$

1987. 2

1988. $\frac{1}{2}$

1989. $\frac{50}{3}$

1990. 1

1991. 1

1992. $\ln\sqrt{3}$

1993. $\frac{1}{9}(2e^3+1)$

1994. $2\sqrt{6}\arctan(\frac{\sqrt{6}}{3})+\ln 100-4$

1995. $2e^2-e+2-\ln 16$

1996. $\ln(\sqrt{10}-1)$

1997. $\log e$

1998. 1

1999. divergent

2000. divergent

2001. 10

2002. divergent

2003. 1

2004. divergent

2005. $\dfrac{1}{k\ln k}$

2006. (a)-(b) positive (c) 0 (d)-(f) negative

2007. (a) -27 (b) 24 (c) 0 (d) 450

2008. 1.408

2009. (a) $y=-15x+3$ (b) $y=\frac{1}{30}(2-x)$ (c) decreasing (d) 0 and 2 (e) 0 (f) 54 (g) -84 (h) 24

2010. $y=\frac{\pi}{2}-x$

2011. 40

2012. (b) $e^2-\ln 4+1$ (c) ≈ 73.564 (d) 19.668 (e) 7.723

2013. 2.899

2024. divergent

2025. divergent

2026. divergent

2027. $\frac{2}{e-2}$

2028. divergent

2029. $\frac{1}{12}$

2030. divergent

2031. divergent

2032. convergent

2033. divergent

2034. convergent

2035. convergent

2036. convergent

2037. divergent

2038. convergent

2039. convergent

2040. $\frac{1}{7}$

2041. $\frac{1}{e^2-1}$

2042. only one is false

2044. (a) $\frac{1}{3}$ (b) no (c) no

2045. (a) 0 (b) no (c) yes

2046. 15

2047. $\frac{172}{333}$

2048. $\frac{1}{4}<x<\frac{5}{4}$, sum is $\dfrac{8}{8-(4x-3)^3}$

2049. $-\frac{\pi}{6}\le x\le\frac{\pi}{6}$

2050. $\frac{4}{3}\le x<\frac{8}{3}$

2051. 20 ft

2052. $\frac{19}{20}$

2053. (a) divergent (b) convergent (c) divergent

2054. (a) $\frac{3}{5}$ (b) 3 (c) ∞

2055. C

2056. C

2057. B

2058. E

Answers to Last Year's Tests

Limits Test

1. C
2. E
3. B
4. B
5. E
6. B
7. A
8. E
9. D
10. B
11. C
12. A
13. C
14. C
15. C

1. a. Since

$$f(x) = \frac{|x|(x-3)}{9-x^2} = \frac{|x|(x-3)}{(3-x)(3+x)} = \frac{-|x|}{3+x},$$

we have that both 3 and -3 are not in the domain; hence, $D = \{x|x \neq \pm 3\}$. The zeros are clearly 0 and 3, but 3 is not in the domain; hence, the only zero is 0.

b.

$$\lim_{x\to 3} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to 3} \frac{-|x|}{3+x} = -\frac{1}{2}.$$

c. Clearly, $x = -3$ is the only vertical asymptote since -3 makes the denominator zero. To find the horizontal asymptotes, simply find the limits at positive and negative infinity:

$$\lim_{x\to\infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to\infty} \frac{-|x|}{3+x} = -1$$

$$\lim_{x\to-\infty} \frac{|x|(x-3)}{9-x^2} = \lim_{x\to-\infty} \frac{-|x|}{3+x} = 1$$

So there are two horizontal asymptotes: $y = 1$ and $y = -1$.

d. Based on the previous parts, it should be easy to see that $x = -3$ is an infinite discontinuity, and therefore is not removable. (Note that $x = 3$ is a hole and so is removable.)

2. a. We have the following values:

x	1	0.5	0.4	0.3
x^x	1	0.707	0.693	0.697

x	0.2	0.1	0.01
x^x	0.725	0.794	0.955

b. Judging from the data in the table, it appears as if both limits are 1. This is confirmed by the graphing calculator.

c. Any answer between 0.697 and 0.707 is fine as long as you justify it using values in the table.

d. The average rate of change is

$$\frac{g(0.4) - g(0.1)}{0.4 - 0.1} = \frac{0.693 - 0.794}{0.3} = -0.337.$$

3. a. This question becomes much simpler if you rewrite F as

$$(a^{-1} - x^{-1})^{-1} = \left(\frac{1}{a} - \frac{1}{x}\right)^{-1} = \left(\frac{x-a}{ax}\right)^{-1} = \frac{ax}{x-a}.$$

Then we can easily see that the domain $D = \{x|x \neq 0, x \neq a\}$ and that there are no zeros.

b. Since $x = a$ is not in the domain, $x = a$ is the vertical asymptote. Since the degree of the numerator is equal to the degree of denominator, we have $y = a$ as the horizontal asymptote. The discontinuities are the infinite discontinuity at $x = a$ and the removable discontinuity at $x = 0$.

c. $\lim\limits_{x\to 0} F(x) = 0$; $\lim\limits_{x\to\infty} F(x) = a$; and $\lim\limits_{x\to a} F(x)$ does not exist.

d. Solve $\frac{6a}{6-a} = 12$ to get $a = 4$.

Derivatives Test

1. D
2. D
3. A
4. C
5. A
6. C
7. D
8. B
9. C
10. D
11. D
12. D
13. D
14. D
15. E

1. a. Taking the derivative implicitly, we have

$$\begin{aligned} y' - y'\sin y &= 1 \\ y'(1-\sin y) &= 1 \\ y' &= \frac{1}{1-\sin y} \end{aligned}$$

b. Vertical tangents have an undefined slope. Hence, we set the denominator of y' equal to zero and solve to get $\sin y = 1$, or $y = \pi/2$. Now we find the x value when $y = \pi/2$:

$$\begin{aligned} \frac{\pi}{2} + \cos\frac{\pi}{2} &= x+1 \\ \frac{\pi}{2} &= x+1 \\ x &= \frac{\pi}{2} - 1 \end{aligned}$$

Hence, the vertical tangent is $x = \frac{\pi}{2} - 1$.

c. We find the second derivative implicitly.

$$y'' = -\frac{y'\cos y}{(1-\sin y)^2}$$

Now plug in the expression for y'.

$$\begin{aligned} y'' &= -\frac{\frac{1}{(1-\sin y)}\cos y}{(1-\sin y)^2} \\ &= -\frac{\cos y}{(1-\sin y)^3} \end{aligned}$$

2. a. The volume is $V = Bh$, where B is the area of the triangular base. Hence, $V = \left(\frac{1}{2}(3)(2)\right)(5) = 15$.

b. By similar triangles, we have

$$\frac{\text{base of triangle}}{\text{height of triangle}} = \frac{2}{3},$$

or $b = \frac{2}{3}h$; so that

$$V = \frac{1}{2}\left(\frac{2}{3}(h)(h)\right)(5) = \frac{5}{3}h^2.$$

When the trough is $\frac{1}{4}$ full by volume, we have $\frac{15}{4} = \frac{5}{3}h^2$, so $h = \frac{3}{2}$ at this instant. Now, we find the implicit derivative with respect to t:

$$\frac{dV}{dt} = \frac{10}{3}h\frac{dh}{dt}$$

and plug in our value of h:

$$\begin{aligned} -2 &= \frac{10}{3}\cdot\frac{3}{2}\cdot\frac{dh}{dt} \\ \frac{dh}{dt} &= -\frac{2}{5} \end{aligned}$$

c. The area of the surface is $A = 5b = 5\cdot\frac{2}{3}h = \frac{10}{3}h$. Finding the implicit derivative and using the value of dh/dt from part (b), we have

$$\begin{aligned} \frac{dA}{dt} &= \frac{10}{3}\cdot\frac{dh}{dt} \\ &= \frac{10}{3}\cdot\frac{-2}{5} = -\frac{4}{3} \end{aligned}$$

3. a. The domain is whatever makes $x^4 - 16x^2 \geq 0$, or $x^2(x^2-16) \geq 0$; thus, we find have either $x = 0$ or $x^2 \geq 16$. The domain is therefore $(-\infty, -4) \cup \{0\} \cup (4, \infty)$.

b. We have

$$\begin{aligned} f(-x) &= \sqrt{(-x)^4 - 16(-x)^2} \\ &= \sqrt{x^4 - 16x^2} = f(x) \end{aligned}$$

so f is even.

c. Observe:

$$\begin{aligned} f'(x) &= \frac{1}{2}(x^4 - 16x^3)^{-1/2}(4x^3 - 32x) \\ &= \frac{2x^3 - 16x}{\sqrt{x^4 - 16x^3}} = \frac{2x(x^2-8)}{|x|\sqrt{x^2-16}} \end{aligned}$$

d. From part (c), we have

$$f'(5) = \frac{10(25-8)}{5\sqrt{25-16}} = \frac{34}{3}$$

so the slope of the normal is $-\frac{3}{34}$.

Applications of Derivatives Test

1. D	**6.** A	**11.** B
2. D	**7.** B	**12.** D
3. D	**8.** E	**13.** D
4. C	**9.** B	**14.** C
5. D	**10.** A	**15.** A

1. a. We have

$$\begin{aligned} v(t) = x'(t) &= 2\pi - 2\pi \sin 2\pi t \\ &= 2\pi(1 - \sin 2\pi t) \end{aligned}$$

b. We have

$$a(t) = v'(t) = x''(t) = -4\pi^2 \cos 2\pi t$$

c. The particle is at rest when $v(t) = 0$:

$$\begin{aligned} 2\pi(1 - \sin 2\pi t) &= 0 \\ \sin 2\pi t &= 1 \\ 2\pi t &= \frac{\pi}{2} \\ t &= \frac{1}{4}, \frac{5}{4}, \frac{9}{4} \end{aligned}$$

d. We find the critical points of $v(t)$ by setting $a(t) = 0$:

$$\begin{aligned} -4\pi^2 \cos 2\pi t &= 0 \\ \cos \pi t &= 0 \\ \pi t &= \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \\ t &= \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots \end{aligned}$$

However, $v\left(\frac{1}{4}\right) = 0$ and $v\left(\frac{3}{4}\right) = 4\pi$ are the only possible maximum values (since all other odd multiples of $\frac{1}{4}$ give values equivalent to these two). Thus, the maximum velocity is 4π.

2. a. The absolute maximum occurs at $x = -1$ because f is increasing on the interval $[-3, -1]$ and decreasing on the interval $[-1, 3]$. The absolute minimum must occur at $x = 1$ or at an endpoint. However, f is decreasing on the interval $[-1, 3]$; therefore, the absolute minimum is at an endpoint. Since $f(-3) = 4 > 1 = f(3)$, the absolute minimum is at $x = 3$.

b. There is an inflection point at $x = 1$ because the graph changes from concave up to concave down (or f'' changes from positive to negative) there.

3. a. We first find critical points:

$$\begin{aligned} f'(x) = 3x^2 - 10x + 3 &= 0 \\ (3x - 1)(x - 3) &= 0 \\ x &= \frac{1}{3} \text{ and } 3 \end{aligned}$$

Since f' is positive for $x < \frac{1}{3}$ and for $x > 3$, the increasing intervals are $\left(-\infty,\ \frac{1}{3}\right)$ and $(3,\ \infty)$.

b. Since $f''(x) = 6x - 10$, the inflection point is $x = \frac{5}{3}$. Thus, since f'' is negative for $x < \frac{5}{3}$, the graph of f is concave down on $\left(-\infty,\ \frac{5}{3}\right)$.

c. From part (a), we know that $x = 3$ gives the minimum value. Hence, we must have $f(3) = 11$:

$$\begin{aligned} f(3) = 3^3 - 5(3^2) + 3(3) + k &= 11 \\ -9 + k &= 11 \\ k &= 20 \end{aligned}$$

Integrals Test

1. E **2.** B **3.** E **4.** B **5.** C

6. D **7.** A **8.** B **9.** C **10.** B

11. C **12.** B **13.** E **14.** D **15.** A

1.

a. We have $T(0) = -15$ and $T(12) = 5$. This gives the system of equations

$$\begin{aligned} -A - B &= -15 \\ -A + B &= 5 \end{aligned}$$

Hence, $A = 5$ and $B = 10$.

b.

$$\begin{aligned} &\frac{1}{10}\int_0^{10} \left(-5 - 10\cos\left(\frac{\pi h}{12}\right)\right)\ dh \\ &\quad = -6.910 \end{aligned}$$

c.

$$\begin{aligned} &\int_6^{10} T(h)\ dh \\ &\quad = \frac{1}{2}[T(6) + 2T(7) + 2T(8) \\ &\qquad + 2T(9) + T(10)] \\ &\quad = \frac{1}{2}[-5 + 2(-2.412) + 2(0) \\ &\qquad + 2(2.071) + 3.66] \\ &\quad = -1.011 \end{aligned}$$

This integral represents the average temperature in degrees Fahrenheit from 6 AM to 10 AM.

d. Since $T(h) = -5 - 10\cos\left(\frac{\pi h}{12}\right)$, we have

$$T'(h) = -\frac{5\pi}{6}\sin\left(\frac{\pi h}{12}\right).$$

2. Differentiating the expression in 1) gives $f''(x) = 2ax + b$. From 2) we have that $f'(1) = 2a+b = 6$ and $f''(1) = 2a+b = 18$. Thus we have a system of equations in a and b that we can easily solve to get $a = 12$ and $b = -6$. Therefore

$$f'(x) = 12x^2 - 6x,$$

and

$$f(x) = \int (12x^2 - 6x)\ dx = 4x^3 - 3x^2 + C.$$

Using 3) we can solve for C:

$$\begin{aligned} 18 &= \int_1^2 f(x)\ dx = x^4 - x^3 + Cx\Big|_1^2 \\ &= 16 - 8 + 2C - (1 - 1 + C) = 8 + C \end{aligned}$$

thus, $C = 10$, and $f(x) = 4x^3 - 3x^2 + 10$.

3.

a.

$$a(t) = v'(t) = -2\pi\cos(2\pi t)$$

b. Set $v(t) = 0$ and solve.

$$\begin{aligned} 1 - \sin(2\pi t) &= 0 \\ \sin(2\pi t) &= 1 \\ 2\pi t &= \frac{\pi}{2} \\ t &= \frac{1}{4}, \frac{5}{4} \end{aligned}$$

c.

$$\begin{aligned} x(t) &= \int v(t)\ dt \\ &= \int (1 - \sin(2\pi t))\ dt \\ &= t + \frac{1}{2\pi}\cos(2\pi t) + C \end{aligned}$$

Since $x(0) = 0$, we have

$$\begin{aligned} 0 &= 0 + \frac{\cos 0}{2\pi} + C \\ 0 &= \frac{1}{2\pi} + C \\ C &= -\frac{1}{2\pi} \end{aligned}$$

Thus, $x(t) = t + \frac{1}{2\pi}\cos(2\pi t) - \frac{1}{2\pi}$.

Applications of Integrals Test

1. A
2. B
3. E
4. A
5. E
6. D
7. A
8. A
9. C
10. D
11. B
12. B
13. D
14. B
15. C

1. a. First, we find the x-coordinates of the intersection points of the two graphs. Set them equal and solve, using your calculator:

$$\begin{aligned} 4e^{-x} &= \tan\left(\frac{x}{2}\right) \\ x &= 1.4786108 \end{aligned}$$

Let $a = 1.4786108$. Thus, the area A is

$$A = \int_0^a \left(4e^{-x} - \tan\left(\frac{x}{2}\right)\right)\, dx = 2.483$$

b. The volume V is

$$\begin{aligned} V &= \pi \int_0^a \left[\left(4e^{-x}\right)^2 - \left(\tan\left(\frac{x}{2}\right)\right)^2\right]\, dx \\ &= 7.239\pi = 22.743 \end{aligned}$$

c. Since the diameter is in R, the length of the radius is $\frac{1}{2}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]$. The area of a semicircle with radius r is $A = \frac{1}{2}\pi r^2$. Hence,

$$\begin{aligned} A &= \frac{\pi}{2}\left(\frac{1}{2}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]\right)^2 \\ &= \frac{\pi}{8}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2. \end{aligned}$$

Therefore, the volume V is

$$\begin{aligned} V &= \int_0^a \frac{\pi}{8}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2\, dx \\ &= 0.755\pi = 2.373 \end{aligned}$$

2. a. You should have segments of zero slope at the three points where $x = 0$. You should have negative slopes with increasing steepness bottom to top at the points where $x = -1$. Finally, you should have positive slopes with increasing steepness from bottom to top and from left to right at the points where $x = 1$ and $x = 2$.

b. You should draw a graph that is concave up, decreasing for $x < 0$, increasing for $x > 0$, and that passes through the point $(0, 2)$.

c. To solve, we separate and integrate:

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy}{2} \\ \int \frac{dy}{y} &= \int \frac{x}{2}\, dx \\ \ln y &= \frac{1}{4}x^2 + C \\ y &= Ce^{x^2/4} \end{aligned}$$

With the initial condition, we find that $C = 2$, so the equation is $y = 2e^{x^2/4}$. Therefore, $y(2) = 2e^{4/4} = 2e = 5.4365$.

3. a. Since $v(1.5) = 1.167 > 0$ the particle is moving up the y-axis.

b. The acceleration is

$$a(t) = v'(t) = \sin(t^2) + 2t^2\cos(t^2)$$

so that $a(1.5) = -2.049 < 0$, which indicates the velocity is decreasing.

c. We have

$$y(t) = \int v(t)\, dt = -\frac{\cos(t^2)}{2} + C$$

and using the initial condition $y(0) = 3$, we find $C = \frac{7}{2}$. Hence,

$$y(t) = \frac{7 - \cos(t^2)}{2}.$$

Therefore, $y(2) = \frac{7-\cos 4}{2} = 3.827$.

d. The total distance is given by

$$\int_0^2 |v(t)|\, dt = 1.173,$$

or

$$\int_0^{\sqrt{\pi}} v(t)\, dt - \int_{\sqrt{\pi}}^2 v(t)\, dt = 1.173.$$

Techniques of Integration Test

1. C
2. B
3. C
4. C
5. E
6. D
7. C
8. C
9. C
10. D
11. B
12. C
13. C
14. C
15. C

1. a. The average value of f from 0 to 3 is

$$\frac{1}{3}\int_0^3 f(x)\ dx = \frac{5-1}{2},$$

and solving for the integral gives

$$\int_0^3 f(x)\ dx = 6.$$

b. Again, the average value of f from 0 to x is

$$\frac{1}{x}\int_0^x f(t)\ dt = \frac{5+f(x)}{2},$$

or

$$\int_0^x f(t)\ dt = \frac{5x+xf(x)}{2}.$$

Using the Fundamental Theorem to differentiate both sides, we have

$$\begin{aligned} f(x) &= \frac{5}{2}+\frac{1}{2}f(x)+\frac{1}{2}xf'(x) \\ 2f(x) &= 5+f(x)+xf'(x) \\ f'(x) &= \frac{f(x)-5}{x}. \end{aligned}$$

c. From part (b), we have a differential equation that can be solved.

$$\begin{aligned} \frac{dy}{dx} &= \frac{y-5}{x} \\ \int \frac{dy}{y-5} &= \int \frac{dx}{x} \\ \ln(y-5) &= \ln x + C \\ y-5 &= Cx \\ y &= Cx+5 \end{aligned}$$

and since $f(3) = -1$, we get that $C = -2$; hence, $y = f(x) = 5-2x$.

2. a.

$$\begin{aligned} R &= \int_1^3 \ln x\ dx = (x\ln x - x)\Big|_1^3 \\ &= 3\ln 3 - 2 = 1.296. \end{aligned}$$

b.

$$V = \pi\int_1^3 (\ln x)^2\ dx = 1.029\pi = 3.233$$

c. We solve $y=\ln x$ for x to get $x=e^y$. When $x=1$, $y=0$, and when $x=3$, $y=\ln 3$. Thus,

$$V = \pi\int_0^{\ln 3} (3-e^y)\ dy$$

3. a.

$$\begin{aligned} \frac{dy}{dx} &= \frac{-xy}{\ln y} \\ \int \frac{\ln y}{y}\ dy &= \int -x\ dx \\ \frac{(\ln y)^2}{2} &= \frac{-x^2}{2}+C \\ (\ln y)^2 &= -x^2+C \\ \ln y &= \pm\sqrt{C-x^2} \\ y &= e^{\pm\sqrt{C-x^2}} \end{aligned}$$

b. We find C.

$$\begin{aligned} y &= e^{\pm\sqrt{C-x^2}} \\ e^2 &= e^{\pm\sqrt{C}} \\ 2 &= \pm\sqrt{C} \\ C &= 4 \end{aligned}$$

so that $y = e^{\pm\sqrt{4-x^2}}$.

c. If $x=2$, then $y=1$ and $\ln y = 0$. This causes the derivative $\dfrac{-xy}{\ln y}$ to be undefined.

Series, Vectors, Parametric, and Polar Test

1. E	**6.** C	**11.** D
2. D	**7.** D	**12.** C
3. B	**8.** D	**13.** E
4. A	**9.** E	**14.** D
5. A	**10.** C	**15.** A

1. a.

$$\begin{aligned}\mathbf{v}(t) &= \left\langle -\frac{3\pi}{4}\sin\frac{\pi t}{4},\ \frac{5\pi}{4}\cos\frac{\pi t}{4}\right\rangle \\ \mathbf{v}(3) &= \left\langle -\frac{3\pi\sqrt{2}}{8},\ -\frac{5\pi\sqrt{2}}{8}\right\rangle \\ ||\mathbf{v}(3)|| &= \sqrt{\frac{18\pi^2}{64}+\frac{50\pi^2}{64}} \\ &= \frac{\pi\sqrt{17}}{4} = 1.031\pi = 3.238\end{aligned}$$

b.

$$\begin{aligned}\mathbf{a}(t) &= \left\langle -\frac{3\pi^2}{16}\cos\frac{\pi t}{4},\ -\frac{5\pi^2}{16}\sin\frac{\pi t}{4}\right\rangle \\ \mathbf{a}(3) &= \left\langle \frac{3\pi^2\sqrt{2}}{32},\ -\frac{5\pi^2\sqrt{2}}{32}\right\rangle \\ &= \langle 0.133\pi^2,\ -0.221\pi^2\rangle \\ &= \langle 1.309,\ -2.181\rangle\end{aligned}$$

c. Since

$$\sin^2\theta + \cos^2\theta = 1,$$

we have, upon solving $x(t)$ and $y(t)$ for the trigonometric terms,

$$\frac{x^2}{3} + \frac{y^2}{5} = 1.$$

2. a. This curve is a 3-petal rose with petal tips at Cartesian coordinates $(\sqrt{3}, 1)$, $(-\sqrt{3}, 1)$, and $(0, -2)$.

b.

$$\frac{1}{2}\int_0^{\pi} (2\sin 3\theta)^2\ d\theta = \pi = 3.142$$

c.

$$\begin{aligned}\frac{dy}{dx} &= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\ &= \frac{6\cos 3\theta\sin\theta + 2\sin 3\theta\cos\theta}{6\cos 3\theta\cos\theta - 2\sin 3\theta\sin\theta} \\ \left.\frac{dy}{dx}\right|_{\theta=\pi/4} &= \frac{1}{2}\end{aligned}$$

3. a.

$$f(x) \approx 5 - 3x + \frac{x^2}{2} + \frac{4x^3}{6}$$

b.

$$g(x) \approx 5 - 3x^2 + \frac{x^4}{2}$$

c.

$$h(x) \approx 5x - \frac{3x^2}{2} + \frac{x^3}{6}$$

d. $h(1) = \int_0^1 f(t)\ dt$, but the exact value cannot be determined since $f(t)$ is only known at $t = 0$ and $t = 1$.