Calculus Workbook

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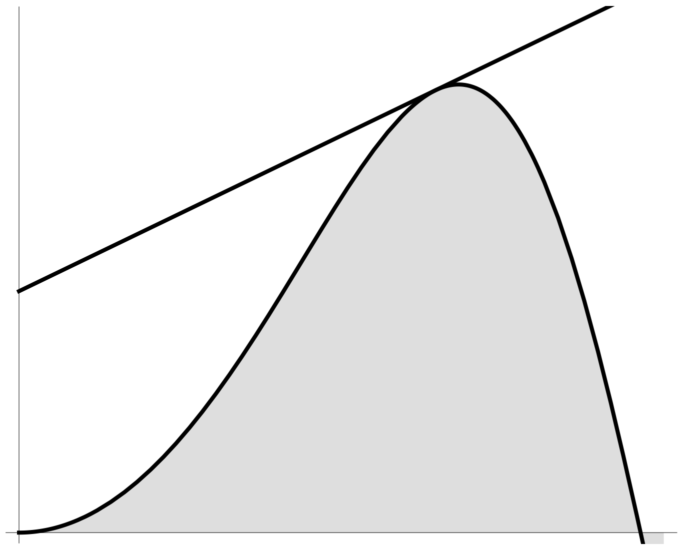


Note to Students: (Please Read) This workbook contains examples and exercises that will be referred to regularly during class. Please purchase or print out the rest of the workbook before our next class and bring it to class with you every day.

1. **To Purchase the Workbook.** Go to *Digi-Type*, the print shop at 1726 E. Cotati Avenue (across from campus, in the strip mall behind the Seven-Eleven). Ask for the workbook for Math 161 - Calculus 1. The copying charge will probably be between \$10.00 and \$20.00. You can also visit the *Digi-Type* webpage (http://www.digi-type.com/) to order your workbook ahead of time for pick-up.

2. **To Print Out the Workbook.** Go to the Moodle page for our course and click on the link "Math 161 Workbook", which will open the file containing the workbook as a .pdf file. BE FOREWARNED THAT THERE ARE LOTS OF PICTURES AND MATH FONTS IN THE WORKBOOK, SO SOME PRINTERS MAY NOT ACCURATELY PRINT PORTIONS OF THE WORKBOOK. If you do choose to try to print it, please leave yourself enough time to purchase the workbook before our next class class in case your printing attempt is unsuccessful.

Table of Contents

Math 161 – Review Examples

Example 1. Let $f(x) = \dfrac{x}{x+1}$.

(a) Solve $f\left(\frac{1}{x}\right) = 2$ for x.

(b) Simplify $f\left(\frac{1}{x}\right)$.

Example 2.

(a) Let $f(x) = x^2$. Calculate and simplify the expression $\dfrac{f(x+h) - f(x)}{h}$, given that $h \neq 0$.

(b) Repeat part (a), but this time let $f(x) = \dfrac{1}{x+1}$.

Example 3. At Bill's Gas 'n' Snacks, it is determined that the demand for gas in a fixed period of time is 100 gallons when the price per gallon is \$1.50, and that the demand decreases by 5 gallons for each 4 cent increase in price.

(a) Find a formula for demand as a function of price; that is, find a formula for f so that $d = f(p)$.

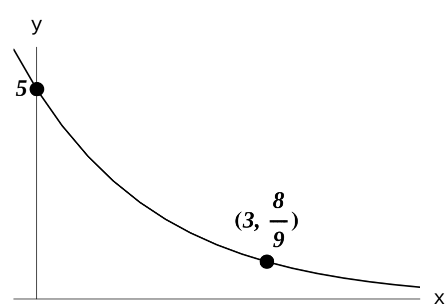
(b) Calculate $f(0)$, and explain its economic significance.

Example 4. Solve for x in each of the following equations.

(a) $5x^8 = 4$ (b) $5 \cdot 8^x = 4$ (c) $\frac{1}{2}x(ax+1)^{-1/2} + \sqrt{ax+1} = 0$

Example 5. Find a formula for $A(x)$, where A is the area of an equilateral triangle of side length x.

Example 6. Find a formula for the exponential function $f(x) = Ca^x$ whose graph is given to the right.



Example 7. Let $f(x) = \dfrac{x^2 - x - 6}{x+2}$ and $g(x) = x - 3$. Are $f(x)$ and $g(x)$ the same function? Explain.

Math 161 – Preliminary Review Problems

Note. The following problems deal with review topics that should already be familiar to you from previous algebra, trigonometry, and precalculus courses. Being able to do problems similar to those below without the aid of a calculator will be important to your success in Math 161.

1. Find a formula for the straight line passing through the point $(2, -4)$ and parallel to $5x - y = 2$.

2. The cost of a box of apples starts at \$20.00 and decreases by 35 cents per day. Find a formula for the cost, C, of the box of apples after t days have passed.

3. Let $g(x) = x^2 + 3x + 4$.

   (a) Calculate $\dfrac{g(a+h) - g(a)}{h}$, where $h \neq 0$, and simplify the result. Your final answer should contain no fractions and no negative exponents.

   (b) Solve the equation $g(2x) = 2$ for x.

4. Let $f(x) = x - \frac{1}{x}$.

   (a) Calculate and simplify $f(x-4)$.

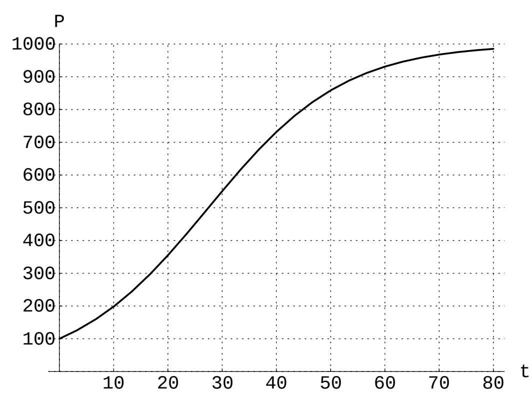
   (b) Calculate and simplify $f(x) - 4$.

   (c) Solve the equation $f(x-4) = 0$ for x.

5. Solve each of the following equations for the the variable x, giving your solutions in **exact** form. For part (b), find all solutions between 0 and 2π **exactly**.

   (a) $10e^{2x} = 7 \cdot 2^{ax}$ (b) $2\sin x + 1 = 0$ (c) $\dfrac{2(a-x)}{3\sqrt[3]{x}} - x^{2/3} = 0$

6. To the right, you are given a graph that shows a function $P = f(t)$, where P is the population of a group of foxes after t months.



   (a) Estimate $f(20)$ and explain what this quantity represents in the context of this problem.

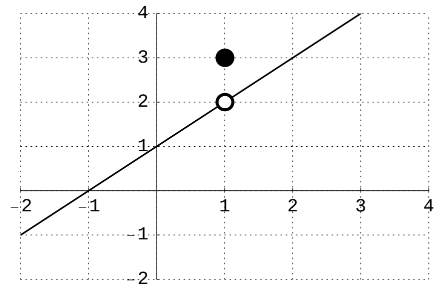
   (b) For $h = 10$, estimate $\dfrac{f(20+h) - f(20)}{h}$ and explain what this quantity represents in the context of this problem.

   (c) Estimate the average rate of change of the fox population during the first 20 months. Include appropriate units with your answer.

Section 2.2 – The Limit of a Function

Definition. We say that $\lim_{x \to a} f(x) = L$, which is read "the limit as x approaches a of $f(x)$ equals L," if we can make $f(x)$ arbitrarily close to L by taking x values sufficiently close to a **but not equal to a.**

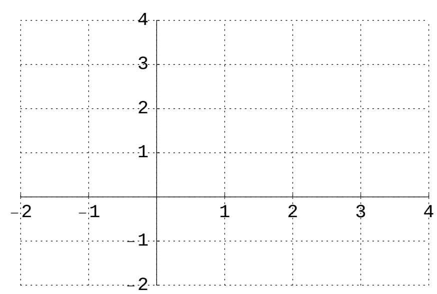
Example 1. Let $f(x) = \begin{cases} x+1 & \text{if} \quad x \neq 1 \\ 3 & \text{if} \quad x = 1 \end{cases}$. Calculate $\lim_{x \to 1} f(x)$ and $\lim_{x \to 2} f(x)$.



Example 2. Let $f(x) = \dfrac{|x-2|}{x-2}$. Fill in the tables below and discuss the existence of $\lim_{x \to 2} f(x)$.

x	$f(x)$
1.9	
1.99	
1.999	
1.9999	

x	$f(x)$
2.1	
2.01	
2.001	
2.0001	



Theorem. $\lim_{x\to a} f(x) = L$ is true if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

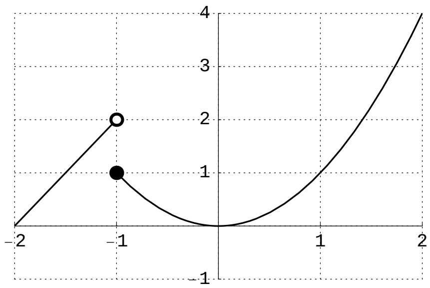
Notes on Limits:

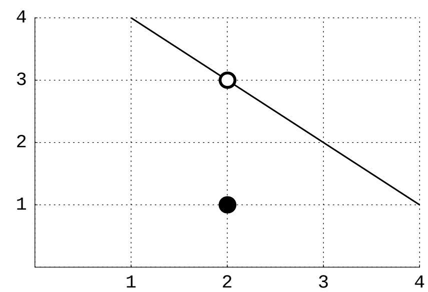
Exercises.

1. Consider the function f and g given below. Then, calculate each of the items that follows.

$$f(x) = \begin{cases} 2x+4 & \text{if } \quad x < -1 \\ x^2 & \text{if } \quad x \geq -1 \end{cases}$$

$$g(x) = \begin{cases} 5-x & \text{if } \quad x \neq 2 \\ 1 & \text{if } \quad x = 2 \end{cases}$$





(a) $f(-1)$

(b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1^-} f(x)$

(d) $g(2)$

(e) $\lim_{x\to 2^+} g(x)$

(f) $\lim_{x\to 2^-} g(x)$

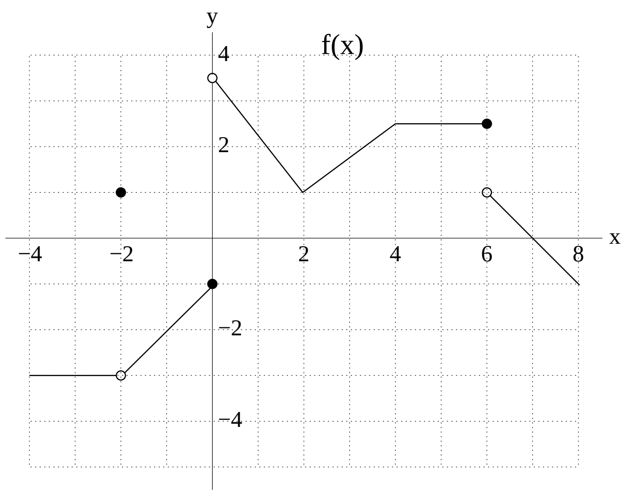
2. (a) Does $\lim_{x\to -1} f(x)$ from Exercise 1 exist? Explain.

   (b) Does $\lim_{x\to 2} g(x)$ from Exercise 1 exist? Explain.

3. Let $f(x) = x^2 - 3$. Is it true that $\lim_{x\to 3} f(x) = f(3)$? Justify your answer.

4. Estimate the value of $\lim_{x\to 0} \dfrac{2^x - 1}{x}$ accurate to 2 decimal places.

5. Use the graph of $f(x)$ given below to estimate the value of each of the following to the nearest 0.1 of a unit.



(a) $f(-2)$
(b) $\lim_{x\to -2} f(x)$
(c) $f(0)$
(d) $\lim_{x\to 0} f(x)$
(e) $\lim_{x\to 0^+} f(x)$
(f) $\lim_{x\to 0^-} f(x)$
(g) $f(2)$
(h) $\lim_{x\to 2} f(x)$
(i) $\lim_{x\to 6^+} f(x)$
(j) $\lim_{x\to 3^-} f(x)$
(k) $\lim_{x\to 3} f(x)$

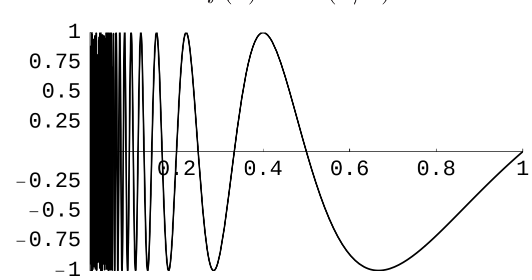
6. Let $f(x) = \sin\left(\dfrac{\pi}{x}\right)$.

(a) Fill in the table below, and then make a guess as to the value of $\lim_{x\to 0^+} f(x)$.

x	0.1	0.01	0.001	0.0001
$f(x)$				

(b) Shown to the right is the graph of the function $f(x) = \sin\left(\dfrac{\pi}{x}\right)$. Now, what do you think about the value of $\lim_{x\to 0^+} f(x)$? Explain.

$f(x) = \sin(\pi/x)$



Section 2.3 – Limit Laws

Properties of Limits

P1. If the graph of f has no breaks or jumps at $x = a$, then

$$\lim_{x \to a} f(x) = \underline{\hspace{5cm}}$$

P2. $\lim\limits_{x \to a} c = \underline{\hspace{10cm}}$

P3. $\lim\limits_{x \to a} [f(x)]^n = \underline{\hspace{10cm}}$

P4. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \underline{\hspace{10cm}}$

P5. $\lim\limits_{x \to a} [f(x) + g(x)] = \underline{\hspace{10cm}}$

P6. $\lim\limits_{x \to a} [f(x) - g(x)] = \underline{\hspace{10cm}}$

P7. $\lim\limits_{x \to a} [c\, f(x)] = \underline{\hspace{10cm}}$

P8. $\lim\limits_{x \to a} [f(x)\, g(x)] = \underline{\hspace{10cm}}$

P9. **(Limits of Quotients)** If $\lim\limits_{x \to a} f(x) = c$ and $\lim\limits_{x \to a} g(x) = d$, then we have the following possibilities for $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$:

(i) If $d \neq 0$, then

(ii) If $c \neq 0$ and $d = 0$, then

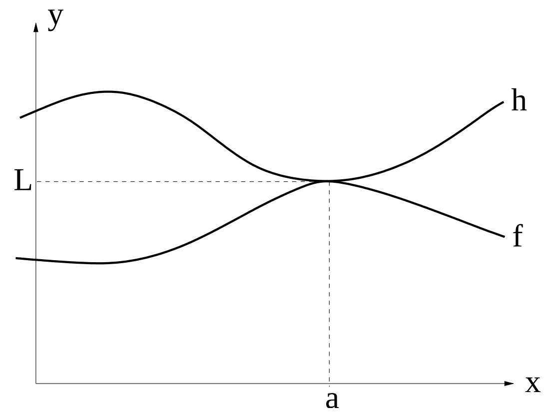
(iii) If $c = 0$ and $d = 0$, then

The Squeeze Theorem. If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

____________________________________.



Exercises

1. Calculate each of the following limits.

   (a) $\lim_{x\to 2}(x^3 + 4x - 5)$

   (b) $\lim_{x\to -1}\left[\frac{x^3 - 2x^2 + 4}{x^2 + 2x + 4} \cdot \sqrt{2x+5}\right]$

   (c) $\lim_{x\to 3}\frac{\sqrt{x}\,(x-3)}{x^2 - 8x + 15}$

(d) $\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h}$

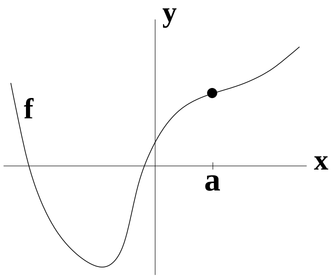
(e) $\lim_{x \to 3} \frac{1}{x-3}$

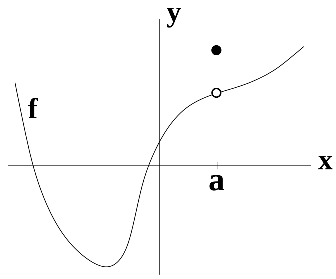
2. Calculate $\lim_{x \to 1} g(x)$, where $g(x) = \begin{cases} x^2+1 & \text{if} \quad x \neq 1 \\ 0 & \text{if} \quad x = 1 \end{cases}$.

3. Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

4. Compute $\lim_{x \to 0} \left[x \sin\left(\frac{\pi}{x}\right) \right]$ using the Squeeze Theorem.

Section 2.4 – Continuity





Definition. We say that a function f is *continuous at a* if $\lim_{x \to a} f(x) = f(a)$. (Graphically, continuity means that f has no breaks or jumps at $x = a$.) In particular, if f is continuous at $x = a$, then the following three things must be true:

1. $f(a)$ must exist.
2. $\lim_{x \to a} f(x)$ must exist.
3. $\lim_{x \to a} f(x)$ must equal $f(a)$.

Example 1. Is $f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if} \quad x \neq -1 \\ 6 & \text{if} \quad x = -1 \end{cases}$ continuous at $x = -1$? Justify your answer.

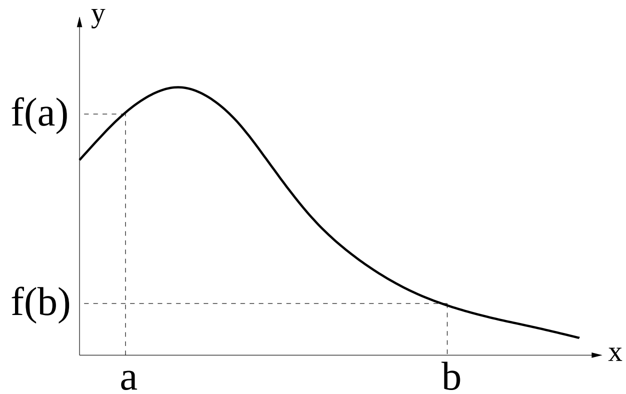
Example 2. Is $f(x) = -\frac{1}{(x-1)^2}$ continuous at $x = 1$?

List of Continuous Functions.

1. Polynomials
2. Rational Functions (on their domains)
3. Root and Power Functions
4. Constant Multiples of Continuous Functions
5. Combinations of Continuous Functions: Sums, Differences, Products, Quotients, Compositions

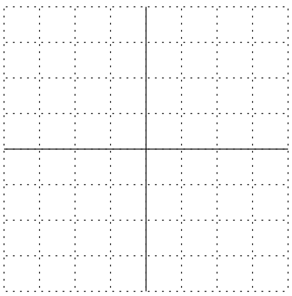
Example 3. Let $f(x) = \begin{cases} x^2 + 4x + 6 & \text{if} \quad x \leq -1 \\ x + 4 & \text{if} \quad -1 < x < 1 \\ 2x - 1 & \text{if} \quad x \geq 1 \end{cases}$. Where is $f(x)$ continuous?

The Intermediate Value Theorem (IVT)
Suppose f is continuous on $[a, b]$, and let N be any number strictly between $f(a)$ and $f(b)$. Then



Exercises.

1. Sketch the graph of a function $f(x)$ such that all of the following are true: $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ both exist, $f(2) = 1$, and $f(x)$ is continuous everywhere EXCEPT at $x = 2$.

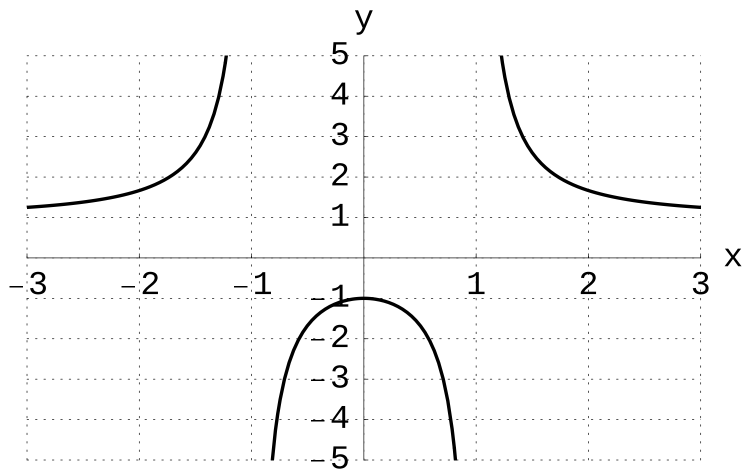


2. Let $f(x) = \begin{cases} cx^2 - 1 & \text{if} \quad x < 2 \\ cx + 2 & \text{if} \quad x \geq 2 \end{cases}$. Find the value of the constant c so that $f(x)$ is continuous for all values of x.

Section 2.5 – Limits Involving Infinity

Preliminary Example. Calculate the following limits.

$$f(x) = \frac{x^2+1}{x^2-1}$$



$\lim_{x\to\infty} f(x) =$ ________

$\lim_{x\to-\infty} f(x) =$ ________

$\lim_{x\to-1^+} f(x) =$ ________

$\lim_{x\to-1^-} f(x) =$ ________

$\lim_{x\to1^+} f(x) =$ ________

$\lim_{x\to1^-} f(x) =$ ________

Definition.

1. f has a *horizontal asymptote* at $y = a$ if ____________________ or if ____________________.

2. f has a *vertical asymptote* at $x = a$ if ____________________ or if ____________________.

Indeterminate Forms. If you encounter any of the following forms when evaluating a limit, the limit *may or may not exist* (i.e. further investigation is required).

(i) $\frac{0}{0}$ (ii) $\frac{\infty}{\infty}$ (iii) $\infty - \infty$ (iv) $0 \cdot \infty$

Exercises.

1. Sketch the graph of a function that satisfies all of the following properties at once.

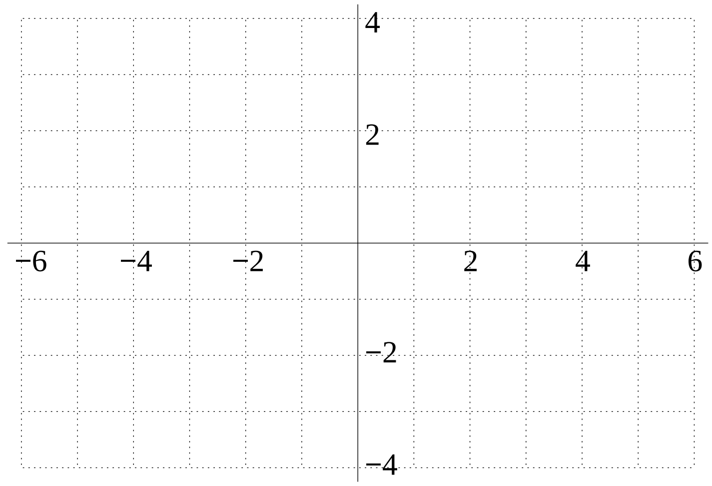
   (a) $\lim_{x\to -2^+} f(x) = \infty$

   (b) $\lim_{x\to -2^-} f(x) = -\infty$

   (c) $\lim_{x\to \infty} f(x) = \infty$

   (d) $\lim_{x\to -\infty} f(x) = 0$

   (e) $\lim_{x\to 3} f(x) = -\infty$



2. Calculate the following limits.

   (a) $\lim_{x\to\infty} \dfrac{7x^3+4x}{2x^3-x^2+3}$

   (b) $\lim_{x\to\infty} (\sqrt{1+x}-\sqrt{x})$

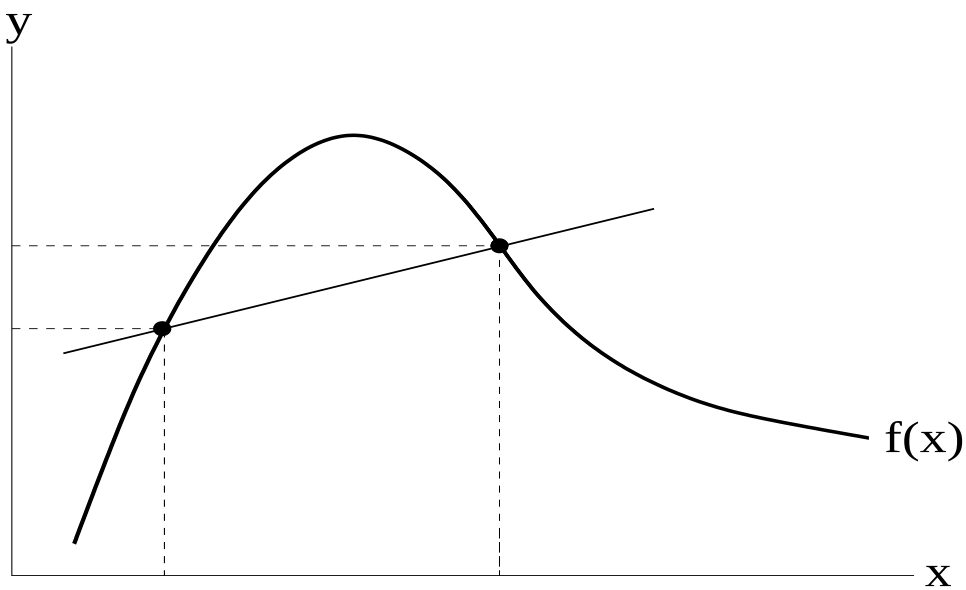
(c) $\lim_{x \to 0^+} \frac{1}{x}$

(d) $\lim_{t \to \pi^+} \csc t$

(e) $\lim_{x \to 0^+} (x^{-\frac{1}{2}} - x^{-\frac{1}{4}})$

3. Let $f(x) = \dfrac{2x^2 + x - 1}{x^2 - 2x - 3}$. Find all vertical and horizontal asymptotes of $f(x)$.

Sections 2.1 & 2.6 – Tangents, Velocities, and Other Rates of Change



Let t = tangent line to $f(x)$ at $x = a$

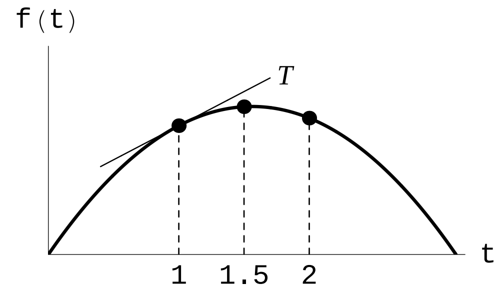
Slope of secant line $\overline{PQ} =$

As $h \longrightarrow 0$, the slope of the secant line $\overline{PQ}$ approaches __________________________. Therefore:

The slope of the tangent line, m, to the curve $y = f(x)$ at the point $(a, f(a))$ is given by

$m =$ or $m =$

Example 1. If a ball is thrown into the air with a velocity of 50 feet per second, its height in feet after t seconds is given by $f(t) = 50t - 16t^2$ (see graph to the right).

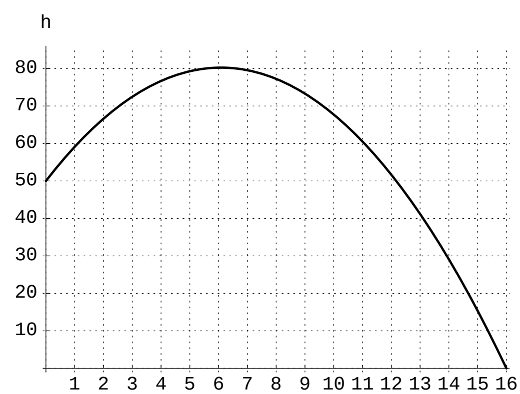


(a) Find the slope of the secant line over the following intervals: (i) $[1, 2]$ (ii) $[1, 1.5]$
Then draw in these lines on the figure to the right.

(b) Use the table below to estimate the slope of T, the tangent line to $f(t)$ at $t = 1$.

Interval	$[1, 1.5]$	$[1, 1.1]$	$[1, 1.01]$	$[1, 1.001]$	$[1, 1.0001]$
Slope of secant line	10	16.4	17.84	17.98	17.9984

Example 2. The graph to the right gives the height of a stone, in meters, thrown upward over the edge of a cliff on the surface of the moon, where t is measured in seconds.



(a) Estimate the average velocity of the stone on the interval $[8, 16]$.

(b) Estimate the instantaneous velocity of the stone at $t = 8$ seconds.

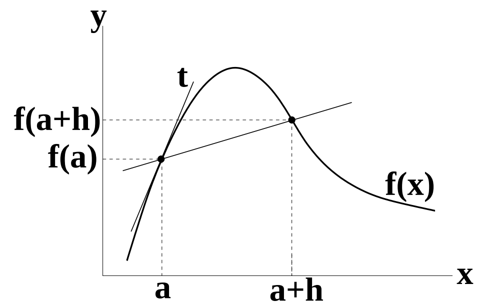
Exercises.

1. Use algebra to find the exact slope of the line T from Example 1 on page 20.

2. Find the slope of the tangent line to $y = \frac{1}{x}$ at $x = 2$.

Section 2.7 – Derivatives

Definition. The <u>derivative</u> of $f(x)$ at $x = a$, is given by

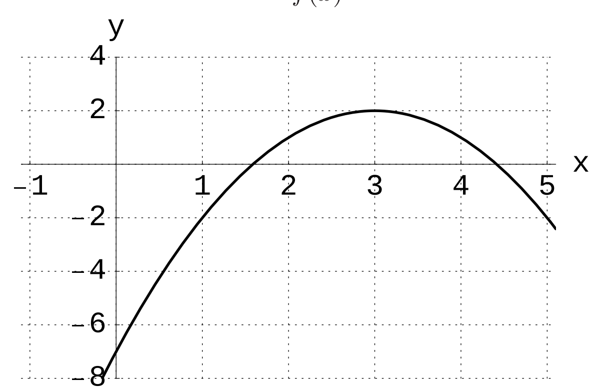


Several Ways To Calculate the Above Limit

Examples and Exercises

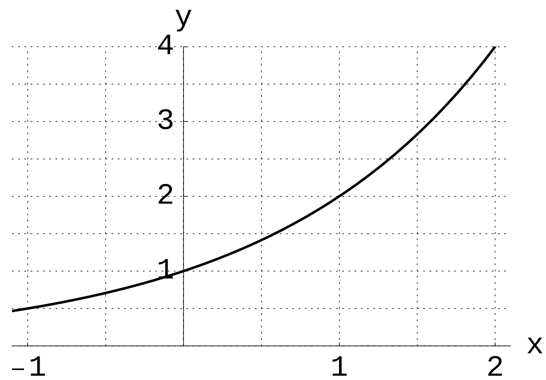
1. Let $f(x) = -x^2 + 6x - 7$. (See graph to the right.) Use the **limit definition** of the derivative to calculate $f'(1)$ and $f'(4)$. What do these two numbers represent geometrically?

$f(x)$



2. Let $f(x) = 2^x$. (See graph to the right). Use the table method to approximate $f'(0)$ accurate to the nearest 0.001.

$f(x)$



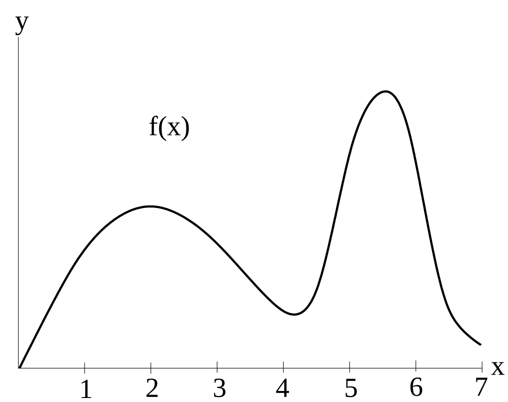
3. The demand for Minnesota Vikings T-Shirts (as a function of price) is given by the function $d(p) = \dfrac{200}{p} - 2$, where p is measured in dollars.

   (a) Calculate $d(10)$ and $d'(10)$ and interpret them in practical terms.

(b) Find the equation of the tangent line to the curve at $(10, 18)$.

4. Consider the graph given to the right. Use the graph to rank the following quantities in order from *smallest* to *largest*:

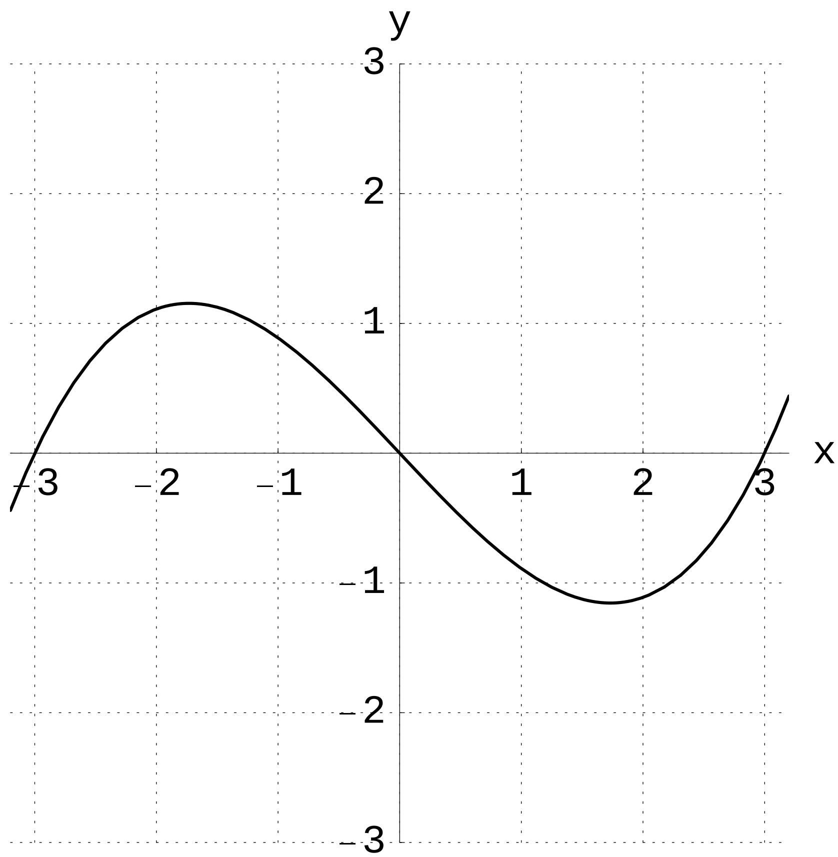
   $0, \qquad f'(0), \qquad f'(1), \qquad f'(3), \qquad f'(6)$



Section 2.8 – The Derivative as a Function

Preliminary Example. Below and to the right, you are given the graph of a function $f(x)$. With a straightedge, draw in tangent lines to $f(x)$ and estimate their slopes from the grid to fill in the table to the left. Then, sketch a rough graph of $f'(x)$.

x	$f'(x)$
-3	
-2	
-1	
0	
1	
2	
3	



Definition. The *derivative function* $f'(x)$ is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

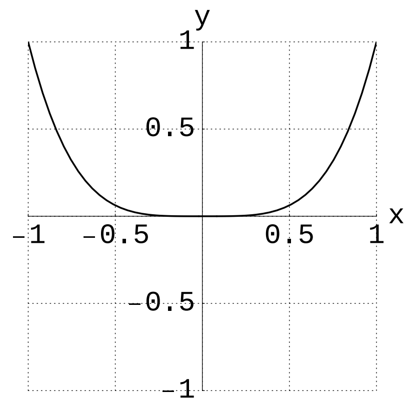
Some Interpretations of the Derivative.

1. $f'(x)$ is the slope of the ____________________ to f at x.
2. If $y = f(x)$, then $f'(x)$ is the instantaneous rate of change of y with respect to x. For example, if $s(t)$ represents the displacement of an object (in meters) after t seconds, then $s'(t)$ represents the ____________ of the object in ____________ at time t.

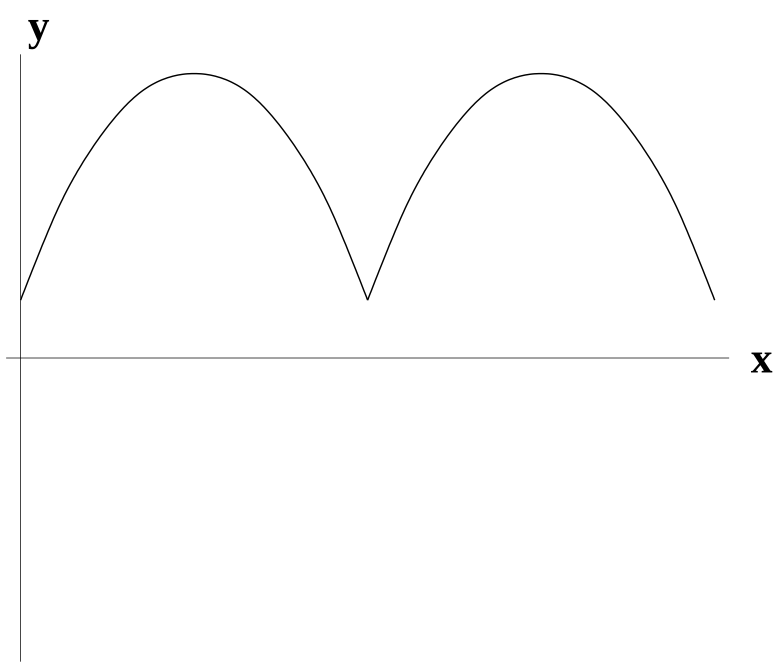
A Point of Terminology. If the derivative of a function exists at a point, then we say that the function is *differentiable* at that point.

Theorem. Let $f(x)$ be a function. If f is differentiable at a number $x = a$, then f is continuous at $x = a$.

Example 1. Calculate a formula for $f'(x)$ if $f(x) = x^4$.
Hint: $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$



Example 2. Given the graph of $y = f(x)$ shown below, sketch a graph of $f'(x)$ on the same set of axes.

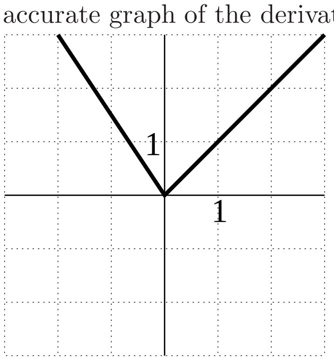


Exercises.

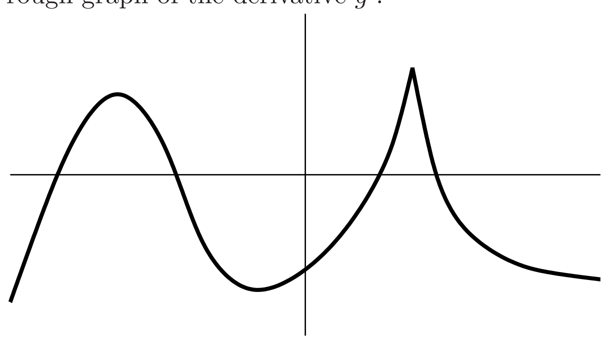
1. Use the **limit definition** of the derivative to find a formula for $f'(x)$ if $f(x) = \sqrt{x}$.

2. Consider the functions f and g that are given below.

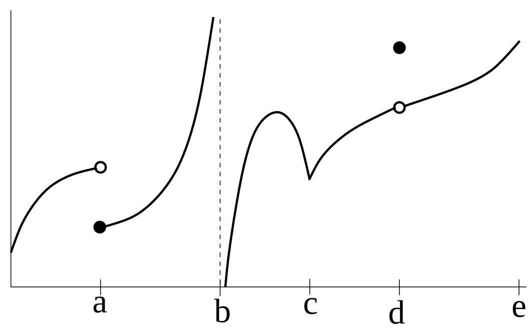
   (a) Given below is the graph of a function f. Sketch an accurate graph of the derivative f'.

   (b) Given below is the graph of a function g. Sketch a rough graph of the derivative g'.

3. Consider the graph of f given to the right.

   (a) On what interval(s) is f continuous?

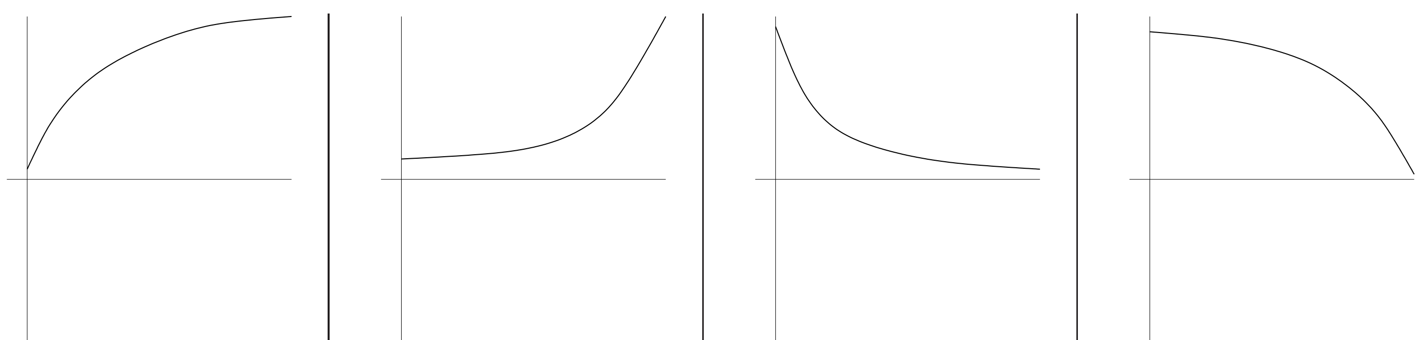
   (b) On what interval(s) is f differentiable?

4. Use the **limit definition** of the derivative to find a formula for $f'(x)$ if $f(x) = \dfrac{1}{2x+1}$.

Section 2.9 – What Does f' Say About f?

Description of $f(x)$	Condition
A function $f(x)$ is called **increasing** on an interval if the graph of $f(x)$ rises from left to right.	
A function $f(x)$ is called **decreasing** on an interval if the graph of $f(x)$ falls from left to right.	
A function $f(x)$ is called **concave up** on an interval if the graph of $f(x)$ looks like part of a right-side up bowl.	
A function $f(x)$ is called **concave down** on an interval if the graph of $f(x)$ looks like part of an upside down bowl.	

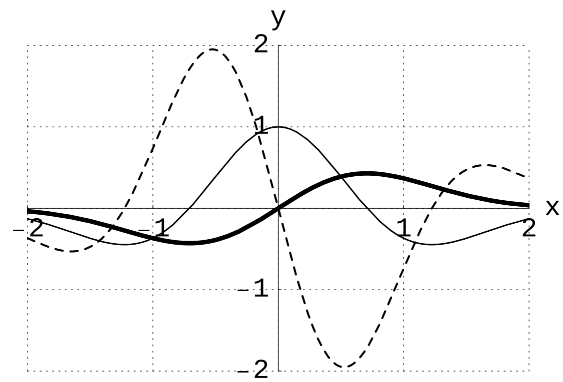
Definition	Picture
We say that a function $f(x)$ has a **local maximum** at $x = a$ if $f(a) \geq f(x)$ for all x values near a.	
We say that a function $f(x)$ has a **local minimum** at $x = a$ if $f(a) \leq f(x)$ for all x values near a.	



Examples and Exercises

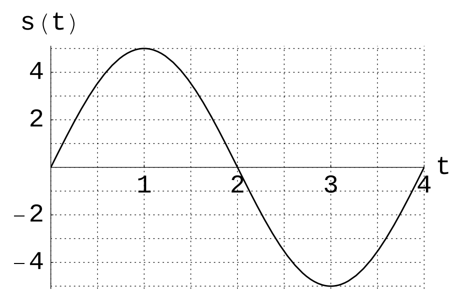
1. In the graph below, you are given a function $f(x)$ and its first and second derivatives. First, decide which is which. Then, answer the following questions.

   (a) On what intervals is $f(x)$ increasing? decreasing?



   (b) On what intervals is $f(x)$ concave up? concave down?

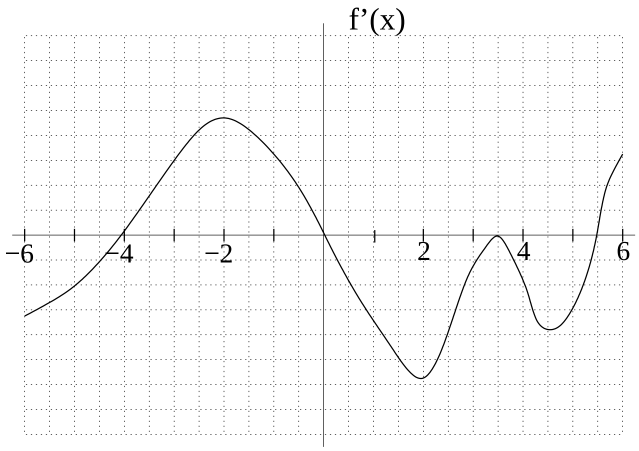
2. The graph to the right gives the position, $s(t)$, of a biker (in feet) from her starting position after t seconds. Positive values of $s(t)$ indicate that she is east of her starting point, and negative values indicate that she is west of her starting point.



   (a) Explain the meaning of $s'(t)$ and $s''(t)$ in the context of this problem.

   (b) Where is $s'(t)$ positive? Where is $s''(t)$ positive? Explain the significance of each answer.

3. Given to the right is $f'(x)$, the graph of the **derivative** of $f(x)$.



   (a) On what interval(s) is the **original function** $f(x)$ increasing? decreasing?

   (b) On what interval(s) is the **original function** $f(x)$ concave up? concave down?

   (c) Where does $f(x)$ have local maximum values? local minimum values?

   (d) Where is $f(x)$ the steepest?

4. In the space to the right, sketch the graph of **one function** $f(x)$ that has all of the following properties.

   (a) $f(0) = 0$.

   (b) $f'(x) > 0$ on $(-\infty, -2)$ and on $(2, \infty)$.
   $f'(x) < 0$ on $(-2, 2)$.

   (c) $f''(x) > 0$ on $(-\infty, -3)$ and on $(0, 3)$.
   $f''(x) < 0$ on $(-3, 0)$ and on $(3, \infty)$.

5. Suppose that $f'(x) = \dfrac{x}{1+x^2}$.

   (a) On what intervals is f increasing? decreasing?

   (b) Estimate the intervals on which f is concave up and the intervals on which f is concave down.

Section 3.1 Information

Alternate Notations for the derivative of $y = f(x)$

1. $f'(x)$ or f'
2. y'
3. $\frac{dy}{dx}$ "the derivative of y with respect to x"
4. $\frac{d}{dx}[f(x)]$

List of Shortcut Formulas

1. $\frac{d}{dx}[c] =$ ______________
2. $\frac{d}{dx}[x^n] =$ ______________
3. $\frac{d}{dx}[cf(x)] =$ ____________________________
4. $\frac{d}{dx}[f(x) + g(x)] =$ ____________________________
5. $\frac{d}{dx}[f(x) - g(x)] =$ ____________________________
6. $\frac{d}{dx}[e^x] =$ ______________

Section 3.1 – Derivatives of Polynomials and Exponential Functions

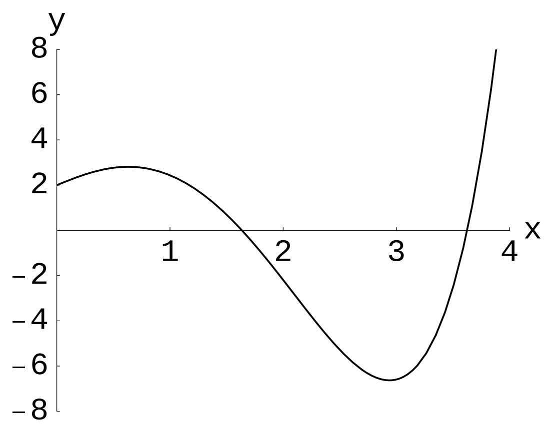
Example 1. Let $f(x) = x^4 - 2x^3$. Find $f'(x)$

Example 2. Let $y = x^2 + x + 4 + \dfrac{1}{x^3}$. Find y'.

Example 3. Calculate $\dfrac{d}{dt}\left(\dfrac{3t^2}{\sqrt[3]{t^2}} + t\sqrt[4]{t}\right)$

Exercises

1. Let $f(x) = 2e^x - 3x^2\sqrt{x}$, whose graph is given to the right. Sketch in the tangent lines to $f(x)$ at $x = 1, x = 2$, and $x = 3$, and calculate their slopes.



2. Let $f(x) = 1 + 2e^x - 3x$.

   (a) At what value(s) of x does $f(x)$ have a horizontal tangent line? Give answer(s) in exact form and as decimal approximations.

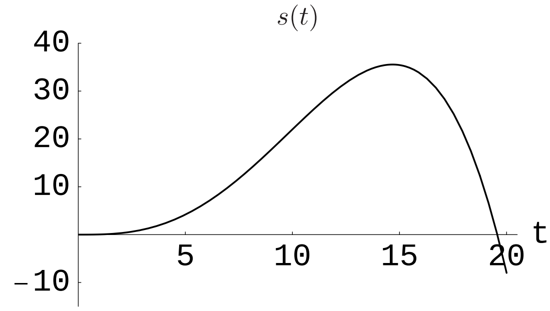
   (b) At what value of x does $f(x)$ have a tangent line parallel to the line $3x - y = 5$?

3. Find the equation of the tangent line to $f(x) = x^3 - 2x^2 + 4x + 1$ at $x = 1$.

4. A jogger is running along the side of a straight east-west highway. Her displacement (in feet) from her home is given by

$$s(t) = 0.045t^3 - 0.0023t^4,$$

where t is measured in seconds. Positive values of $s(t)$ indicate that she is east of home, while negative values of $s(t)$ indicate that she is west of home.



(a) Explain **in words** what the first and second derivatives of displacement represent. What are their units in this problem?

(b) Calculate the displacement, velocity, and acceleration of the runner at $t = 5$ seconds. Include the proper units for each.

(c) Is the runner's velocity ever zero? Is her acceleration ever zero? If so, calculate these times.

(d) What is the runner's velocity at the instant that she passes her home running west?

Section 3.2 – Product and Quotient Rule

$$\frac{d}{dx}\left[f(x)\,g(x)\right] =$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] =$$

Example 1. Calculate $\dfrac{d}{dx}\left[x^2 \cdot x^3\right]$.

Example 2. Let $f(x) = (1 + x^2 - 3x^4)(x^7 - 8x^4)$. Calculate $f'(x)$.

Example 3. Let $y = \dfrac{x^2}{c + xe^x}$. Calculate $\frac{dy}{dx}$.

Example 4. Let f be a differentiable function. If $y = x^4 f(x)$, find formulas for y' and y''.

Exercises

1. Find the derivative of each of the following functions. You may assume that a, b, c, and d are constants.

   (a) $f(x) = (x^2 - \sqrt{x}) \cdot \mathrm{e}^x$

   (b) $g(x) = \dfrac{ax+b}{cx+d}$

2. Find the equation of the tangent line to $y = \dfrac{1}{1 + x + x\mathrm{e}^{x}}$ at $x = 0$.

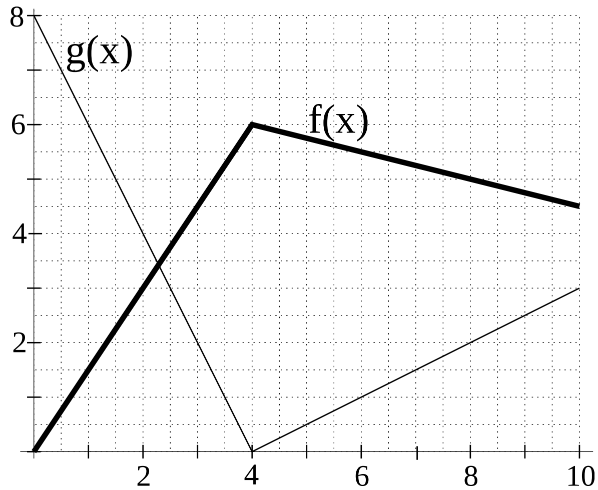
3. Let f and g be the functions given by the graph below, and define the following new functions:

$$u(x) = f(x)g(x),$$

$$v(x) = \frac{f(x)}{g(x)},$$

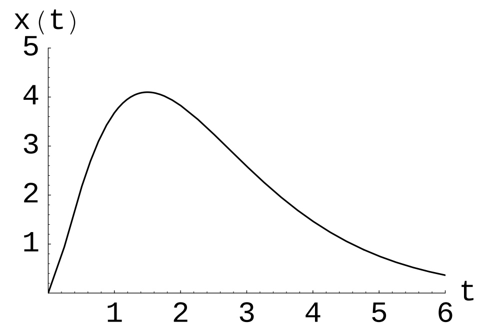
$$w(x) = e^x f(x).$$

Calculate $u'(1), v'(8)$, and $w'(8)$ Make it **clear** which answer is which.



Section 3.3 – Natural/Social Sciences

1. A particle moves along the x-axis, its position at time t given by $x(t) = 10t^{3/2}/e^t$, where t is measured in seconds and $x(t)$ is measured in meters (see the graph to the right).



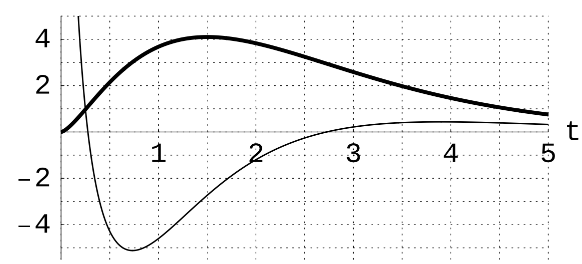
(a) Fill in the entries in the table below and interpret their meaning in the context of this problem.

t	0	1	2	3	4
$x(t)$					

(b) Find a simplified formula for the velocity, $v(t)$, of the particle.

(c) When is the particle moving to the left? When is it moving to the right?

(d) Given to the right is a graph of the displacement of the particle, $x(t)$, together with its acceleration (the lighter of the two graphs). Use the graph to estimate when the particle's speed is increasing.



2. The volume of water, V (in gallons) in a leaking tank after t minutes is given by $V = 10000(1 - \frac{t}{50})^2$. Find the rate at which water is draining from the tank after $t = 10, 20,$ and 30 minutes.

3. Consider a blood vessel with radius $R = 0.008$ cm and length $k = 2$ cm. Suppose that blood viscosity is $\eta = 0.027$ and that $P = 4000$ dynes per square centimeter is the pressure difference between the ends of the vessel. The law of laminar flow states that the velocity, v, of the blood is given, in cm/sec, by

$$v = \frac{P}{4\eta k}(R^2 - r^2),$$

where r is the distance away from the center of the blood vessel, in centimeters.

(a) Where is v the greatest, in the center of the blood vessel or near the vessel wall?

(b) Calculate $\frac{dv}{dr}$, the *velocity gradient*, at $r = 0.002$ cm, and interpret the result in the context of this problem.

Sections 3.4 and 3.5 – The Chain Rule and Trigonometric Functions

The Chain Rule. If $y = f(u)$ is differentiable and $u = g(x)$ is differentiable, then

1. $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2. $$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Note:

Example 1. Let $y = (3x^2 - 2x - 5)^4$. Find $\frac{dy}{dx}$.

Example 2. Let $y = \sqrt{2e^x + 1}$. Find $\frac{dy}{dx}$.

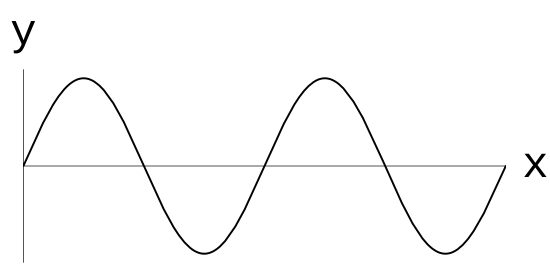
Example 3. Find $\dfrac{d}{dx}\, e^{x^3 - 2x^2 + 1}$.

Example 4. Let $f(t) = \dfrac{t^4}{\sqrt[3]{t^2+1}}$. Find $f'(t)$.

Example 5. Suppose f and g are differentiable functions with the values given in the table to the right. Find $h'(2)$ if $h(x) = f(g(x))$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	7	4	3	6
4	2	-1	5	8

Exploration. Given the the right is a graph of the function $f(x) = \sin x$. On the same set of axes, sketch rough graphs of $f'(x)$ and $f''(x)$, labeling both of them. Then, use your graphs to guess formulas for as many of the derivatives in the box on page 45 as you can.



Derivatives of Trigonometric Functions

$\frac{d}{dx}\sin x =$	$\frac{d}{dx}\cos x =$
$\frac{d}{dx}\tan x =$	$\frac{d}{dx}\cot x =$
$\frac{d}{dx}\sec x =$	$\frac{d}{dx}\csc x =$

Examples and Exercises

1. Find the equation of the tangent line to $f(x) = (x^2 + 1)^{-1}$ at $x = 1$.

2. Use the fact that $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ to prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

3. Find $\frac{d}{dx}(\sin(x\mathrm{e}^x))$.

4. Find the derivative of each of the following functions.

(a) $f(x) = \sqrt{\dfrac{x^2+9}{x+3}}$

(b) $f(t) = \sin(t^2)$

(c) $y = x\mathrm{e}^{-x^2}$

(d) $g(t) = \tan^2 t$

(e) $f(x) = \sqrt{\cos(\sin^2 x)}$

5. The population, P (in billions) is well-modeled by the equation $P = 6e^{0.013t}$, where t is the number of years after the beginning of 1999. First, predict the population of the world in 2009. Then, predict the rate at which the world's population will be growing in 2009. Include units with your answers.

6. Let f be a differentiable function, and let $g(x) = [f(\sqrt{x})]^3$.

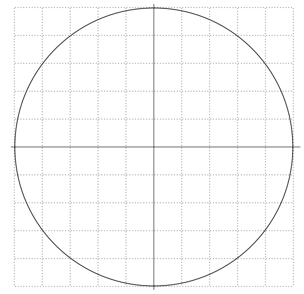
   (a) Calculate $g'(x)$.

   (b) Use the values in the table below to calculate $g'(4)$.

x	$f(x)$	$f'(x)$
2	1	-2
4	-3	4

Section 3.6 – Implicit Differentiation

Preliminary Example. Consider the circle $x^2 + y^2 = 25$, whose graph is shown to the right. Find a formula for $\frac{dy}{dx}$ without solving for y first.



Example 1. Let $x^2 + 3xy + 4y^2 = 4$. Find y'.

Example 2. Let $\frac{1}{x} = y^3 + x\mathrm{e}^y$. Find y'.

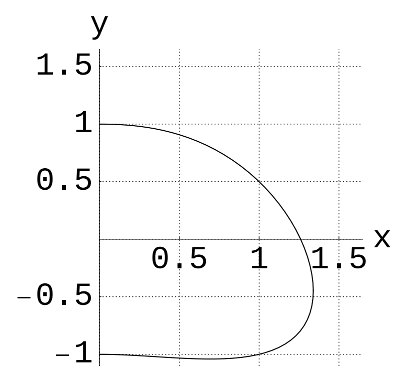
Exercises

1. Find y' if $4\cos x \cos y = 3y$.

2. Find y' if $\mathrm{e}^{xy} + y^2 = 2x$.

3. Find the equation of the tangent line to the curve $x^3 + x^2y + 2y^2 = 2$ at the point $(1,\ 0.5)$. Then, sketch this line on the diagram to the right.

$x^3 + x^2y + 2y^2 = 2$



4. Use the new derivative shortcut formulas below to calculate the derivatives of the following functions.

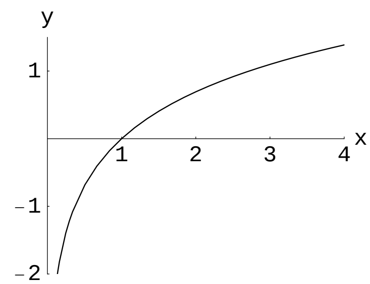
$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$$

(a) $f(x) = \sin^{-1}(x^2)$

(b) $g(x) = [\tan^{-1}(\sqrt{x})]^2$

Section 3.7 – Derivatives of Logarithmic Functions

Preliminary Example. Given to the right is a graph of $f(x) = \ln x$. First, sketch a rough graph of $f'(x)$, and then use implicit differentiation to derive a formula for $f'(x)$.



New Derivative Formulas

1. $\dfrac{d}{dx}[\ln x] =$ ____________________________
2. $\dfrac{d}{dx}[a^x] =$ ____________________________ (where a is constant)
3. $\dfrac{d}{dx}[\log_a x] =$ ____________________________ (where a is constant)

Summary of Derivatives Involving Exponents

Let a be a constant.

4. $\dfrac{d}{dx}[x^a] =$ ____________________________ (Variable Base, Constant Exponent)
5. $\dfrac{d}{dx}[a^x] =$ ____________________________ (Constant Base, Variable Exponent)
6. $\dfrac{d}{dx}\left[f(x)^{g(x)}\right] =$ ____________________________ (Variable Base, Variable Exponent)

Example. Find the derivative of each of the following:

(a) $y = [\ln(\sin x)]^3$ (b) $y = 2^{x^2} \ln x$ (c) $y = x^x$ (d) $y = (\sin x)^{x^2}$

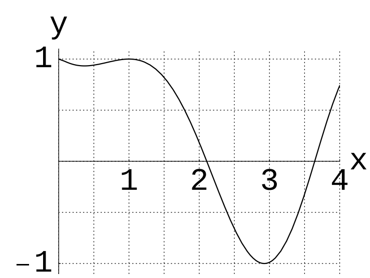
Exercises

1. Let $s(t) = \sin(t^2)$ be the displacement of a particle (in feet) from the origin after t seconds. Find the velocity and acceleration of the particle.

2. For each of the following functions, find y'. Then, find the slope of the tangent line to the curve at $x = 1$. Compare with the provided graphs to see if your answer is reasonable.

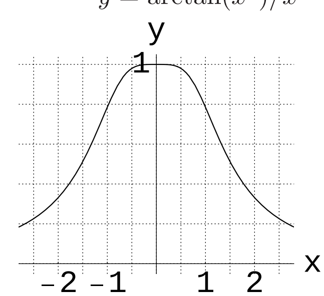
   (a) $y = \cos(x \ln x)$

$y = \cos(x \ln x)$



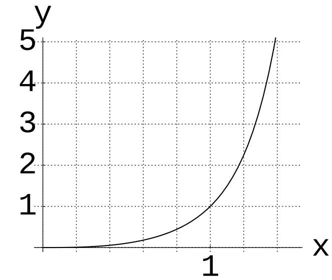
   (b) $y = \dfrac{\arctan(x^2)}{x^2}$

$y = \arctan(x^2)/x^2$



(c) $y = x^{x^2+3}$

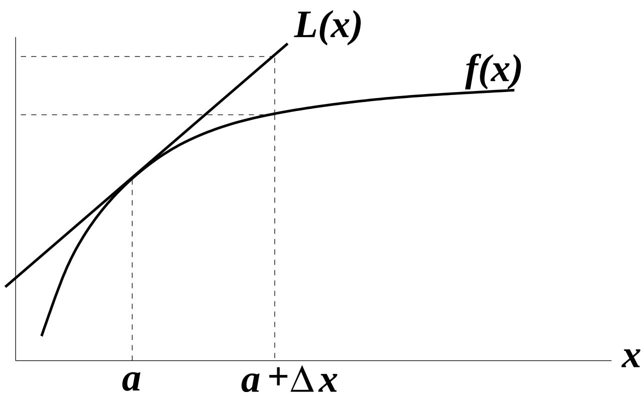
$y = x^{x^2+3}$



(d) $e^{xy} + 1 = x + y^2 e$ (**Hint:** When $x = 1$, $y = 1$.)

3. For the curves in parts (b) and (c) of problem 2, find the equation of the tangent line at $x = 1$.

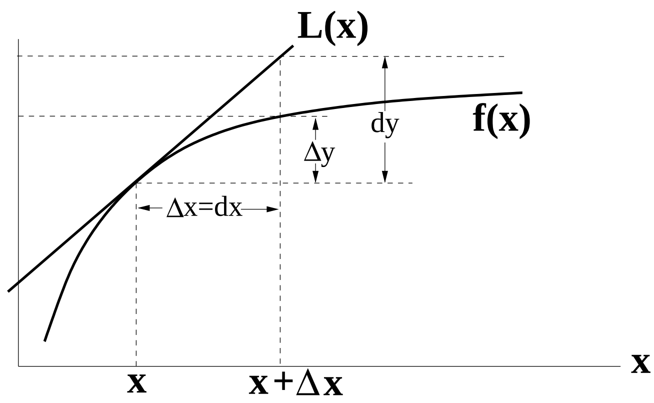
Section 3.8 – Differentials and Linear Approximations



Assume: $f(a)$ and $f'(a)$ are either given or can be easily computed.

Example 1. Let $f(x) = \ln x$. Find the linearization of $f(x)$ at $x = 1$ and use it to approximate $\ln(1.1)$ and $\ln(1.2)$. Also, fill in the table to the right.

x	$\ln x$	$L(x)$	Error $(f(x) - L(x))$
1.1			
1.2			



Assume: $f(x)$ and $f'(x)$ are either given or can be easily computed.

Example 2. Let $y = \sqrt{x}$.

(a) Find the differential dy.

(b) Evaluate dy and Δy if $x = 1$ and $\Delta x = dx = 0.1$.

Exercises

1. Let $f(x) = x^{10}$. Find the linearization of f at $x = 1$ and use it to approximate 0.95^{10}.

2. Let $y = \sqrt{1 + 2x}$.

   (a) Find the differential dy.

   (b) Evaluate dy and Δy if $x = 0$ and $dx = \Delta x = 0.1$.

3. Let $f(x) = e^{2x}$.

   (a) Find the linear approximation of f at $x = 0$.

   (b) If we want to calculate $e^{0.01}$, what value of x would we need to substitute into the function f?

   (c) Use your linear approximation from part (a) to estimate the value of $e^{0.01}$.

Section 4.1 – Related Rates (Taken from *Stewart*)

Problem 1. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cubic centimeters per second. How fast is the radius of the balloon increasing when its diameter is 50 centimeters?

Problem 2. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 feet per second, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ radians?

Recommended Steps in a Related Rates Problem

1. After drawing a diagram (if possible) and giving variable names to all quantities that are functions of time, write down your GIVEN INFORMATION and what you are trying to FIND using derivative notation.

2. Write an EQUATION that relates the various quantities in the problem. If necessary, eliminate one of the variables (as we do in problem 3 on page 63).

3. Take the DERIVATIVE of both sides of the equation WITH RESPECT TO t.

4. AFTER you have taken the derivative of both sides, SUBSTITUTE IN THE GIVEN INFORMATION AND SOLVE the equation for what you want to find.

CAUTION! Do NOT plug in the given information into your equation until AFTER you have taken the derivative of both sides.

Problem 3. A water tank has the shape of an inverted circular cone with base radius 2 meters and height 4 meters. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 meters deep.

Problem 4. A kite 100 feet above the ground moves horizontally at a speed of 8 feet per second. At what rate is the angle between the string and the horizontal decreasing when 200 feet of string have been let out?

Section 4.2 and 4.3 Information

Definition. Let $f(x)$ be a function whose domain is D.

1. An **absolute maximum** of $f(x)$ on D occurs at $x = c$ if $f(c) \geq f(x)$ for all x in D.
2. An **absolute minimum** of $f(x)$ on D occurs at $x = c$ if $f(c) \leq f(x)$ for all x in D.
3. A **local maximum** of $f(x)$ occurs at $x = c$ if $f(c) \geq f(x)$ for all x in some open interval containing c.
4. A **local minimum** of $f(x)$ occurs at $x = c$ if $f(c) \leq f(x)$ for all x in some open interval containing c.

Note. The *maximum* or *minimum value* of a function is a value of the *output* variable. The *place* where a maximum or minimum value *occurs* is a value of the *input* variable.

Definition. A **critical number** of a function f is a number c in the domain of the function f such that **either** $f'(c) = 0$ **or** $f'(c)$ does not exist.

Extreme Value Theorem. If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

Review from Section 2.9:

$f'(x) > 0$	$f'(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
$f(x)$ is increasing	$f(x)$ is decreasing	$f(x)$ is concave up	$f(x)$ is concave down

First Derivative Test. Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a ______________________________ at c.

(b) If f' changes from negative to positive at c, then f has a ______________________________ at c.

(c) If f' does not change sign at c, then f has no local maximum or minimum at c.

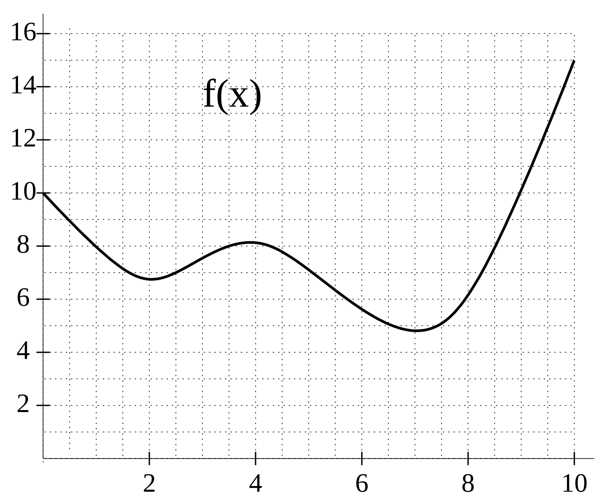
Second Derivative Test. Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a ______________________________ at $x = c$.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a ______________________________ at $x = c$.

Section 4.2 – Maximum and Minimum Values

Preliminary Example. For the function f given below, locate all local and absolute maxima and minima on the interval $[0, 10]$.



General Rule. To find the absolute maximum and the absolute minimum of a continuous function on a closed interval (i.e., an interval that contains its endpoints), compare the output values of the function at the following locations:

1.
2.

Example. Find the absolute maximum and absolute minimum value of $f(x) = x + \frac{3}{x}$ on the interval $[1, 4]$.

Exercises

1. Find the absolute maximum and absolute minimum value of $f(x) = x\mathrm{e}^{-x^2}$ on the interval $[0, 2]$.

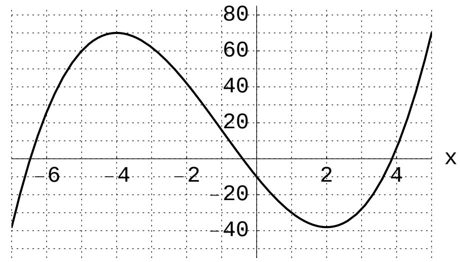
2. (Taken from *Hughes-Hallett, et. al.*) When you cough, your windpipe contracts. The speed, v, at which the air comes out depends on the radius, r, of your windpipe. If 0.5 cm is the normal (rest) radius of your windpipe, then for $0 \leq r \leq 0.5$, the speed is given by $v = (150 - 300r)r^2$. What value of r maximizes the speed?

3. Give an example of a function f and a domain such that the function has no absolute maximum value and no absolute minimum value on that domain.

Section 4.3 – Derivatives and the Shapes of Curves

Preliminary Example. Consider the graph of $f(x)$ given to the right. Label any points on the graph that look like local maxima or local minima. Also label any inflection points (just estimate as best as you can).

$f(x) = x^3 + 3x^2 - 24x - 10$



(a) Make a first-derivative sign chart to determine where f' is positive and where it is negative. Then, list all the information about f that you can based on the chart.

(b) Make a second-derivative sign chart to determine where f'' is positive and where it is negative. Then, list all the information about f that you can based on the chart.

Exercises

1. Let $f(x) = 3x^5 - 10x^3 + 20$.

   (a) Find all critical numbers of $f(x)$, as well as all intervals on which $f(x)$ is increasing/decreasing.

   (b) Find all local maximum and local minimum values of $f(x)$.

   (c) Find all intervals on which $f(x)$ is concave up/down, and list all inflection points of $f(x)$.

2. Let $f(x) = x^{2/3}(4-x)^{1/3}$.

   (a) Given that $f'(x) = \dfrac{8-3x}{3x^{1/3}(4-x)^{2/3}}$, find the intervals on which f is increasing/decreasing.

   (b) Given that $f''(x) = \dfrac{-32}{9x^{4/3}(4-x)^{5/3}}$, find the intervals on which f is concave up/concave down.

   (c) Find all local maxima, local minima, and inflection points of f.

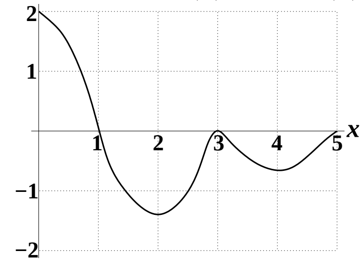
3. Let $f(x) = \dfrac{x^2 + 8}{2x}$.

   (a) Find all local maximum and local minimum values of f.

   (b) Find all intervals on which f is concave up/concave down. Does f have any inflection points?

4. Given to the right is the graph of the DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION f.

Graph of $f'(x)$, NOT $f(x)$



   (a) What are the critical points of f?

   (b) Where is f increasing? decreasing?

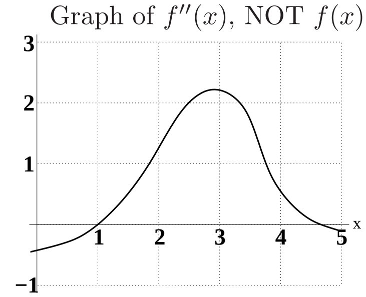
   (c) Does f have any local maxima? If so, where?

   (d) Does f have any local minima? If so, where?

   (e) Where is f concave up? concave down?

5. Given to the right is the graph of the SECOND DERIVATIVE of a function. Use this graph to help you answer the following questions about the ORIGINAL FUNCTION f.

Graph of $f''(x)$, NOT $f(x)$



   (a) Where is f concave up? concave down?

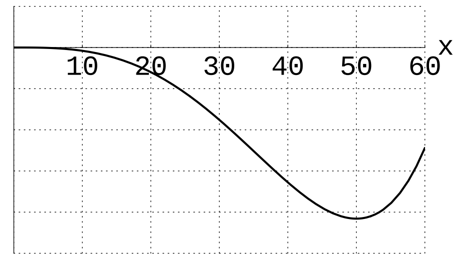
   (b) Does f have any inflection points? If so, where?

6. Consider the function $f(x) = \frac{1}{5}x^5 - \frac{2501}{3}x^3 + 2500x$ on the interval $[0, 60]$.

   (a) Use the fact that $f'(x) = (x-50)(x+50)(x-1)(x+1)$ to find the x and y coordinate of all local maxima and local minima of f on the interval $[0, 60]$.

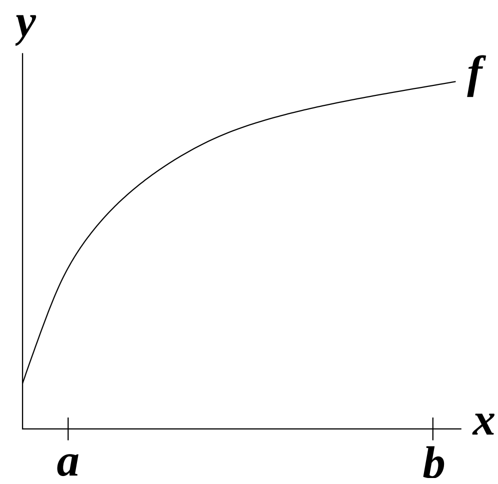
   (b) Explain why the local minimum of f is visible on the graph to the right, but the local maximum of f is not visible.

Graph of $f(x)$



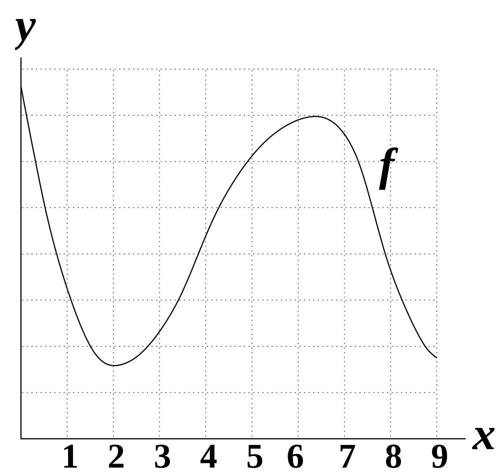
The Mean Value Theorem

Mean Value Theorem. If f is a differentiable function on the interval $[a, b]$, then there exists a number c between a and b such that



Example. Consider the function f shown to the right.

(a) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[0, 9]$. Also draw in the corresponding secant line and tangent lines.



(b) Estimate the value(s) of c that satisfy the conclusion of the Mean Value Theorem on the interval $[2, 6]$.

Section 4.5 – Indeterminate Forms and L'Hospital's Rule

Preliminary Example. Use a table of values to estimate the value of $\lim_{x\to 0} \frac{\sin(2x)}{3x}$. Is it possible to find the exact value of the limit?

The 5 basic indeterminate forms: $\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \frac{-\infty}{\infty}, \quad \frac{\infty}{-\infty}, \quad \frac{-\infty}{-\infty}$

L'Hospital's Rule. Suppose $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Then if $\lim_{x\to a} \frac{f(x)}{g(x)}$ has one of the five indeterminate forms listed above, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Caution: To use L'Hospital's rule, your limit MUST BE in one of the 5 basic indeterminate forms.

Other Indeterminate Forms

- Products: $0 \cdot \infty$ or $0 \cdot (-\infty)$
- Differences: $\infty - \infty$
- Powers: $0^0, \quad \infty^0, \quad$ or 1^∞

Caution: L'Hospital's rule can only be applied to a limit involving a RATIO $\frac{f(x)}{g(x)}$. Therefore, when you encounter one of the "other" indeterminate forms in the box above, you will first need to rewrite your expression as a ratio BEFORE you can apply L'Hospital's rule.

Example. Calculate $\lim_{x\to\infty} \frac{e^x}{x^3}$.

Exercises

Directions. Calculate each of the following limits **exactly**.

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

2. $\displaystyle\lim_{x\to \infty} \frac{\ln x}{x}$

3. $\displaystyle\lim_{x\to 0} \frac{x}{e^x}$

4. $\displaystyle\lim_{x\to \infty} \frac{\ln(\ln x)}{\sqrt{x}}$

5. $\lim_{x \to 1^+} (x-1)\tan\left(\frac{\pi}{2}x\right)$

6. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

Section 4.6 – Optimization Problems

Preliminary Example. A farmer with 2400 feet of fencing wants to fence off a rectangular field that borders on a straight river. She needs no fence along the river. What are the dimensions of the field that has the largest area?

Key Steps in Optimization Problems

(A) SETUP

- Draw and label a diagram, if possible.
- Identify what is given and what you need to find.

(B) EQUATION

- Find an equation involving the the quantity to be maximized or minimized. If the equation has more than one input variable, use the given information to eliminate a variable.

(C) FIND ABSOLUTE MAXIMUM OR MINIMUM

- Remember to look at critical points and at endpoints (if there are any). Be aware that in many applied problems, there won't be endpoints.

Examples and Exercises (Adapted from *Stewart*)

1. A closed rectangular box with a square base and top must have a volume of 32000 cubic centimeters. If it costs 4 cents per square centimeter to construct the bottom and the top and 2 cents per square centimeter to construct the sides, find the dimensions that minimize the cost of making the box. Also find the minimum cost.

2. Consider the portion of the graph of the parabola $y = c - x^2$ that lies above the x-axis, where c is a positive constant. Find the largest possible area of a rectangle with its base on the x-axis and its other two vertices on the parabola. Note that your answer will be in terms of the constant c.

3. A metal can manufacturer needs to build cylindrical cans with volume 300 cubic centimeters. The material for the side of a can costs 0.03 cents per cm^2, and the material for the bottom and top of the can costs 0.06 cents per cm^2. What is the cost of the least expensive can that can be built?

4. Find two numbers x and y whose difference is 100 and whose product is a minimum.

5. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

6. Suppose that we have a 10-foot-long piece of wire. We are allowed to do one of three things with the wire: (i) bend the entire wire into a circle, (ii) bend the entire wire into a square, or (iii) cut the wire in one place and bend one of the two pieces into a circle and the other into a square. Out of all possible scenarios, we would like to know the **maximum** and the **minimum** total area that we could obtain.

   (a) If we bend the entire wire into a circle, calculate the area of that circle.

   (b) If we bend the entire wire into a square, calculate the area of that square.

   (c) If we cut the wire in the middle, and bend half of the wire into a circle and half into a square, what is the combined area of the circle and the square? Draw a picture to illustrate this situation.

   (d) Now, let's generalize this process. Suppose that we cut off a piece of the wire x feet long to bend into a circle, and that we bend the remaining piece into a square. Find, in terms of x, (i) a formula for the area of the circle, and (ii) a formula for the area of the square.

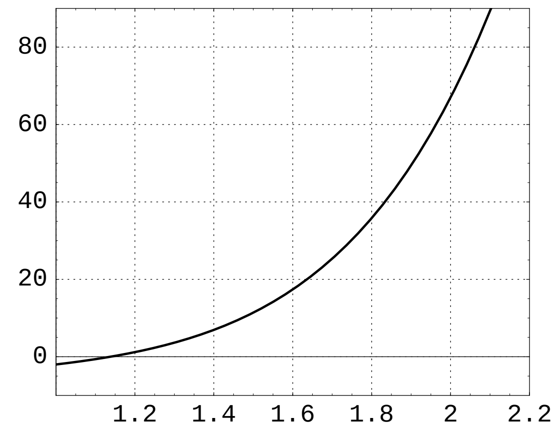
(e) Write down a formula for the combined area, A, of your circle and square in the previous part of the problem, as a function of x. Then, find the absolute maximum and absolute minimum value of your function.

Section 4.8 – Newton's Method

Goal. Find a solution to the equation

$$x^6 + 6x - 9 = 0.$$

$f(x) = x^6 + 6x - 9$



Newton's Method. Let x_1 be an initial guess for a solution to the equation $f(x) = 0$. Then

$$x_{n+1} = -\frac{f(x_n)}{f'(x_n)} + x_n.$$

Notes.

1. By repeatedly using the above formula, we generate a sequence of numbers $x_2,\ x_3,\ x_4, \ldots$.
2. Our hope is that, as we calculate more and more numbers using the above formula, the numbers will get closer and closer to an actual solution to the equation $f(x) = 0$. When this does happen, we say that the sequence *converges*.

Example. Use Newton's Method to approximate the solution to the equation $x^6 + 6x - 9 = 0$ that lies between $x = 1$ and $x = 2$.

Section 4.9 – Antiderivative Problems

Preliminary Example. $\dfrac{d}{dx}x^2 =$

Definition. A function F is called an *antiderivative* of f on an interval I if ________________.

$f(x)$	Antiderivative of $f(x)$
$2x$	
$3x^2$	
$\cos x$	
e^x	
$\sec^2 x$	
k (constant)	
0	
$\sin x$	
x^2	
x^3	
x^n	
x^{-1}	

Check:

1. Find the most general antiderivative of $f(x) = x^2 - \sqrt[3]{x} - 5\cos x$.

2. Suppose that $f''(x) = \frac{2}{x^2} + 6x$.

   (a) Find a general formula for $f(x)$.

   (b) Find $f(x)$ such that $f(1) = 4$ and $f'(1) = 0$.

3. Find f if $f''(x) = 6x,\ f(1) = 2,\ f(2) = 18$.

4. A stone is dropped from the top of the KSDN radio tower in Aberdeen, South Dakota, which stands 600 meters above the ground. You may assume that stone experiences a constant acceleration of 9.8 meters per second squared toward the ground.

   (a) Find the distance of the stone above the ground at time t.

   (b) How long does it take the stone to reach the ground?

   (c) With what velocity does the stone strike the ground?

   (d) Find the distance of the stone above the ground if, instead of being dropped, it is thrown upward with an initial velocity of 5 meters per second.

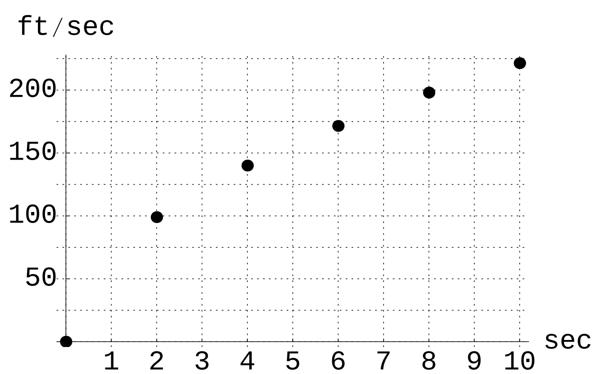
5. A stone was dropped off a cliff and hit the ground with a speed of 150 feet per second. What is the height of the cliff? You may assume that the stone experiences a constant acceleration of 32 feet per second squared toward the ground.

6. What constant deceleration is needed to stop a braking car in 4 seconds if its initial speed is 70 feet per second?

Section 5.1 – Areas and Distances

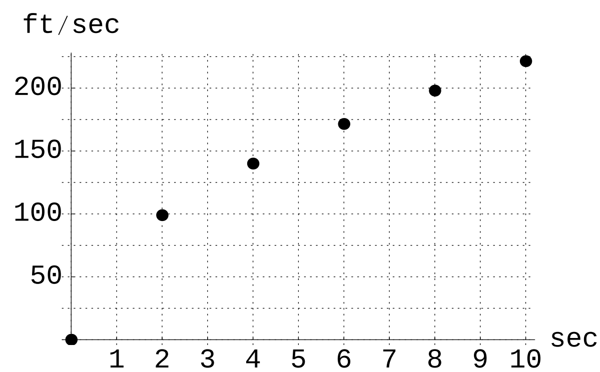
Example 1. The following data is gathered as a small plane travels down the runway toward takeoff. How far did the plane travel in the 10 second period? (Give a range of values.)

time (sec)	0	2	4	6	8	10
velocity (ft/sec)	0	99	140	171.5	198	221.4



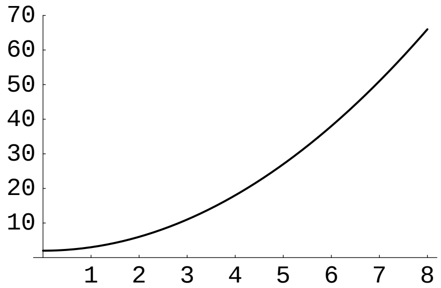
Example 2. Suppose that the same small plane as in Example 1 is traveling toward takeoff, but that now, we are given the velocity of the plane every second (as shown in the table below). Give a new range of values representing the distance that the plane could have traveled, and illustrate your estimates with a new rectangles diagram.

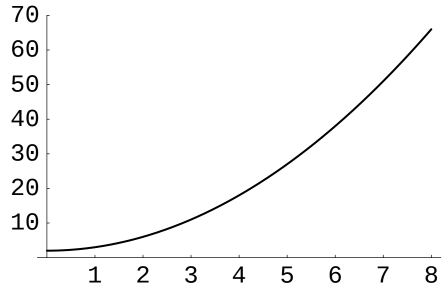
time (sec)	0	1	2	3	4	5	6	7	8	9	10
velocity (ft/sec)	0	75	99	125	140	162	171.5	182	198	215	221.4



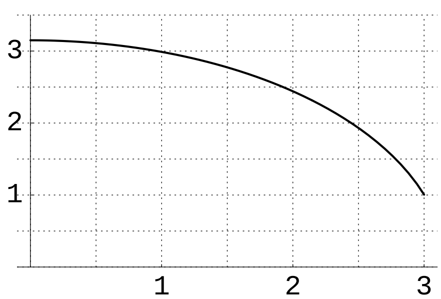
Section 5.2 – The Definite Integral

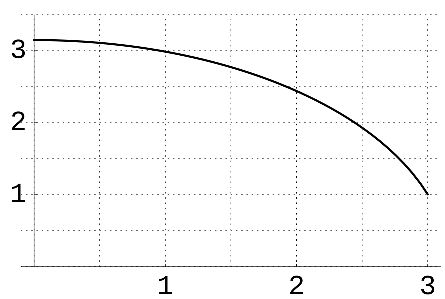
Example 1. Use a left sum and a right sum, with $n = 4$, to estimate the area under the curve $f(x) = x^2 + 2$ on the interval $0 \leq x \leq 8$.



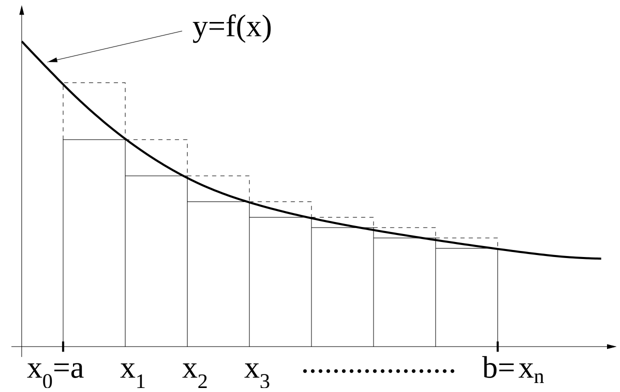


Example 2. Use a left sum and a right sum, with $n = 3$, to estimate the area under the curve $g(x)$ (shown below) on the interval $0 \leq x \leq 3$.





Goal. Describe, in general, a way to find the *exact* area under a curve.



$n =$ number of rectangles

$\Delta x =$ width of one rectangle

Left Sum $=$

Right Sum $=$

Definition. The *definite integral* of $f(x)$ from a to b is the limit of the left and right hand sums as the number of rectangles approaches infinity. We write

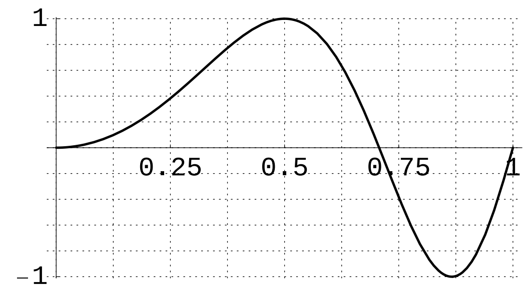
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}[\text{Right-hand sums}] = \lim_{n\to\infty}[\text{Left-hand sums}]$$

Example 3. Use the results of Examples 1 and 2 to give your best estimate of $\int_0^8 (x^2+2)\,dx$ and $\int_0^3 g(x)\,dx$. Then, explain what these integrals represents geometrically.

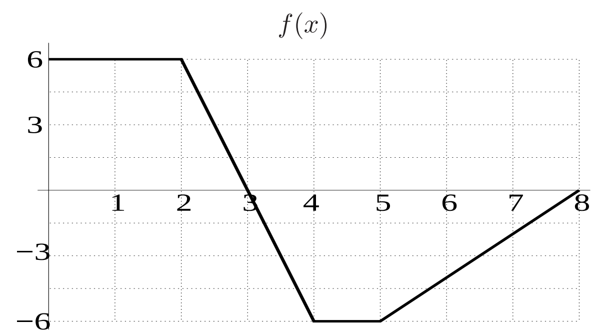
Example 4. Use a "middlesum" with $n = 4$ to estimate the value of

$$\int_0^1 h(x)\,dx,$$

where the graph of h is given to the right.



Example 5. Let f be the graph of the function shown to the right. Calculate each of the integrals that follow *exactly*.



$$\int_0^1 f(x)\,dx \quad =$$

$$\int_0^2 f(x)\,dx \quad =$$

$$\int_0^3 f(x)\,dx \quad =$$

$$\int_4^5 f(x)\,dx \quad =$$

$$\int_5^8 f(x)\,dx \quad =$$

$$\int_0^5 f(x)\,dx \quad =$$

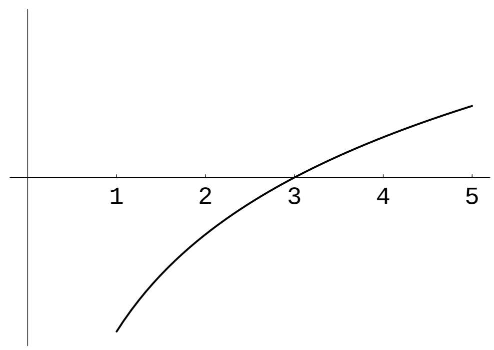
$$\int_2^4 f(x)\,dx \quad =$$

Example 6. Let f be the function defined on $0 \le x \le 12$, some of whose values are shown in the table below. Estimate the value of $\int_0^{12} f(x)\,dx$.

x	0	3	6	9	12
$f(x)$	20	10	5	2	1

Section 5.2 – Integration Practice

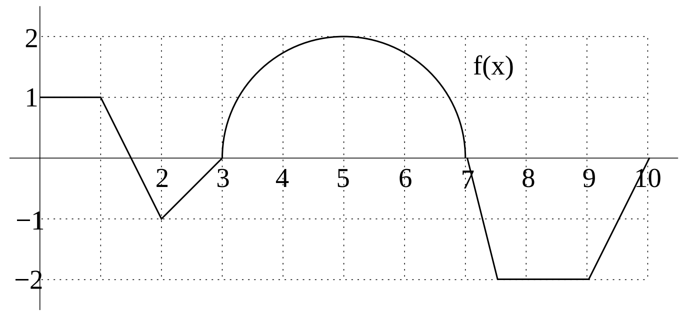
1. Given to the right is the graph of the function $f(x) = \ln(x/3)$.



   (a) Using the picture above, decide whether $\int_1^5 \ln(x/3)\,dx$ is positive or negative.

   (b) Use a middlesum with $n = 4$ to estimate the value of $\int_1^5 \ln(x/3)\,dx$.

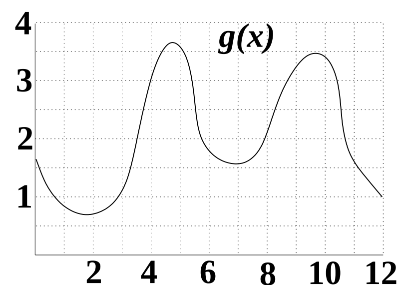
2. Given to the right is the graph of a function $y = f(x)$. Use the graph to calculate the value of each of the following integrals. You may assume that the portion of the graph of $f(x)$ that looks like a semicircle really is a semicircle.



(a) $\displaystyle\int_0^2 f(x)\,dx$ (b) $\displaystyle\int_3^7 f(x)\,dx$

(c) $\displaystyle\int_2^7 f(x)\,dx$ (d) $\displaystyle\int_5^8 f(x)\,dx$

3. Use the graph of the function $g(x)$ to the right to estimate the value of $\displaystyle\int_0^{12} g(x)\,dx$. Use a leftsum, a rightsum, and a middlesum. Use $n = 6$ rectangles in all 3 cases.

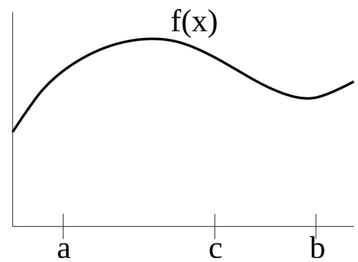


Properties of the Integral

1. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx =$ ______________________

2. $\displaystyle\int_a^a f(x)\,dx =$ ______________________

3. $\displaystyle\int_b^a f(x)\,dx =$ ______________________



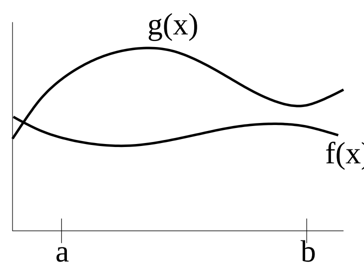
4. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx =$ ____

5. $\displaystyle\int_a^b cf(x)\,dx =$ ____ (if c is constant)

Comparison Properties

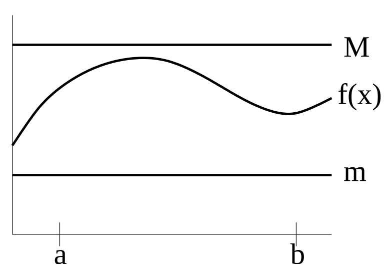
6. If $f(x) \leq g(x)$ for all $a \leq x \leq b$, then

____.



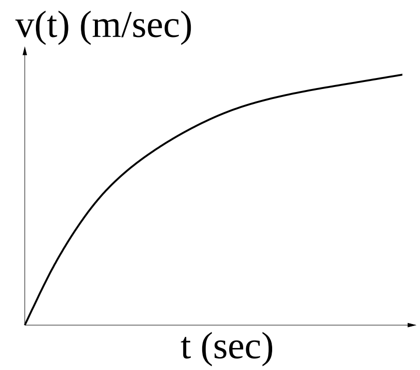
7. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

____.



Section 5.3 – Evaluating Definite Integrals

Preliminary Example. A small plane starts at the end of a runway and accelerates toward takeoff. Let $s(t)$ represent the position of the plane (in meters) relative to its starting point after t seconds. Let $v(t)$ represent the velocity of the plane (in meters per second) after t seconds. Write down 2 expressions that represent the *change in position* of the plane between $t = a$ and $t = b$ seconds.



Net Change Principle. Let $f'(x)$ be continuous on the interval $[a, b]$. Then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

In other words, the integral of a rate of change gives the net change.

Evaluation Theorem. Let $f(x)$ be continuous on the interval $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is an antiderivative of f.

Notes.

1. The two theorems above say exactly the same thing.
2. The evaluation theorem gives us a way to calculate the exact value of $\int_a^b f(x)\,dx$ provided that we can find an ____________________ of f.

Example 1. Calculate $\int_0^1 x\,dx$.

Example 2. Calculate $\int_0^{2\pi} \sin x \, dx$.

Example 3. Calculate $\int_1^4 \left(\frac{10 - 5x}{\sqrt{x}} \right) dx$.

Definite Versus Indefinite Integrals

Definite Integral: $\int_a^b f(x) \, dx$

Indefinite Integral: $\int f(x) \, dx$

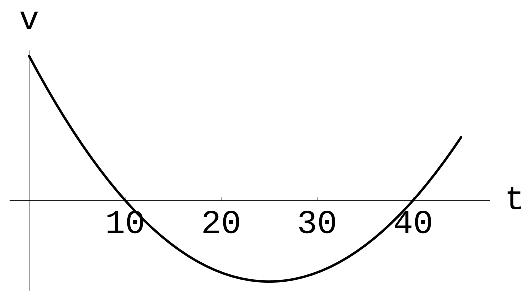
Note.

Exercises.

1. Evaluate the following definite integrals.

   (a) $\displaystyle\int_{-5}^{1} (2e^x - 4\cos x)\,dx$ (b) $\displaystyle\int_{-2}^{-1} \left(\frac{3}{t^4} + \frac{2}{t}\right) dt$ (c) $\displaystyle\int_{0}^{2} x(x^2+1)^2\,dx$

2. The velocity of a biker riding along a straight east/west highway is given by $v(t) = \frac{1}{50}t^2 - t + 8$ feet per second, where t is measured in seconds. (See graph to the right.) Positive velocities indicate eastward travel, while negative velocities indicate westward travel.



   (a) Evaluate and interpret $\int_0^{10} v(t)\, dt$ in the context of this problem.

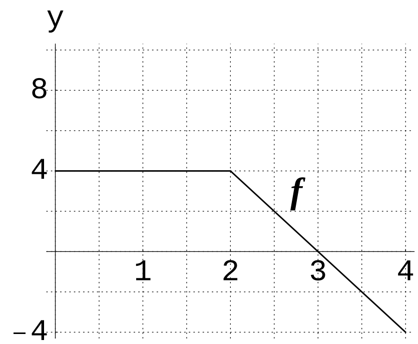
   (b) Evaluate and interpret $\int_0^{40} v(t)\, dt$ in the context of this problem.

   (c) Calculate the total distance traveled by the biker between $t = 0$ and $t = 40$ seconds.

Section 5.4 – Fundamental Theorem of Calculus

Preliminary Example. Let f be the function shown in the diagram to the right, and define

$$g(x) = \int_0^x f(t)\,dt.$$



Example. Calculate $\dfrac{d}{dx}\displaystyle\int_1^x t^2\,dt$ and $\dfrac{d}{dx}\displaystyle\int_2^x \cos t\,dt$.

Fundamental Theorem of Calculus. If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

is a(n) ____________________ of f. In other words, $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt =$ ____________________.

Example. Find a formula for the derivative of each of the following functions.

(a) $g(x) = \displaystyle\int_3^x \sin(t^2)\,dt$ (b) $h(x) = \displaystyle\int_x^\pi \frac{1}{y^3+2}\,dy$ (c) $k(x) = \displaystyle\int_{-2}^{\sqrt{x}} \sqrt{1+3t^2}\,dt$

Exercises.

1. Find a formula for the derivative of each of the following functions.

   (a) $f(x) = \displaystyle\int_3^x (2t+1)\,dt$

   (b) $g(x) = \displaystyle\int_x^2 \frac{u^2}{u^2+1}\,du$

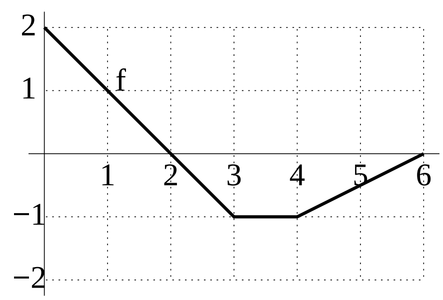
(c) $h(x) = \int_1^{x^2} \mathrm{e}^{-t^2}\, dt$

2. Let f be the function whose graph is given to the right. Define

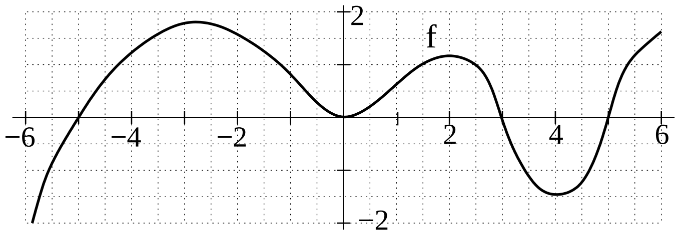
$$F(x) = \int_0^x f(t)\, dt.$$

Fill in the entries in the table that follow:

x	1	2	3	4	5
$F(x)$					
$F'(x)$					
$F''(x)$					



3. Let $F(x) = \int_{-3}^{x} f(t)\,dt$, where $f(x)$ is the function whose graph is given below.



(a) What are the critical points of $F(x)$?

(b) Where is $F(x)$ increasing? decreasing?

(c) Locate all places where $F(x)$ has a local maximum or a local minimum, and make it clear which are which.

(d) Where is $F(x)$ concave up? concave down?

(e) Estimate $F(0)$ using a method of your choice.

Section 5.5 – The Substitution Rule

Preliminary Example. Evaluate $\int 2x\sqrt{1+x^2}\,dx$.

Example 1. Evaluate $\int \frac{\sin x}{\cos^2 x}\,dx$.

Example 2. Evaluate $\displaystyle\int_0^2 x\,\mathrm{e}^{x^2}\,dx$.

Example 3. Evaluate $\displaystyle\int_1^{\mathrm{e}} \frac{\ln x}{2x}\,dx$.

The Substitution Rule. Under "nice" conditions (see formulas 4 and 5 in the text for the precise statement of the conditions), we can choose $u = g(x)$ to obtain the following formulas:

1. $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$
2. $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$

Comments on the Substitution Rule:

1. When making a substitution, the goal is to transform an existing integral into an easier integral that we can then evaluate.
2. While there is no set method that will always work when it comes to choosing your u, here are some guidelines that will often work:
   (a) Choose "u" so that "du" appears somewhere in your integrand.
   (b) Choose "u" to be the "inside" portion of some composite function.

Justification of the Substitution Rule:

Exercises

For problems 1-10, evaluate each of the following integrals. Keep in mind that using a substitution may not work on some problems. For one of the definite integrals, it is not possible to find an antiderivative using any method. Once you have figured out which one this is, use the midpoint rule with $n = 4$ to approximate the value of the integral.

1. $\displaystyle\int \sin x \cos x \, dx$

2. $\displaystyle\int x \sin (x^2 + 5) \, dx$

3. $\displaystyle\int_1^3 \frac{e^{1/x}}{x^2}\,dx$

4. $\displaystyle\int \tan x\,dx$

5. $\displaystyle\int_0^2 e^{x^2}\,dx$

6. $\displaystyle\int_1^4 \frac{2x-3}{x^2}\,dx$

7. $\displaystyle\int_0^2 \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\,dx$

8. $\displaystyle\int_0^2 \frac{\mathrm{e}^x+1}{\mathrm{e}^x}\,dx$

9. $\displaystyle\int \frac{1}{1+4x^2}\, dx$

10. $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\, dx$

11. Let a be any positive real number.

    (a) Explain why $a^x = \mathrm{e}^{x \ln a}$

    (b) Use part (a) together with a u-substitution to evaluate $\displaystyle\int a^x\, dx$.