CALCULUS
EXPLORATION OF THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

$$\frac{d}{dx}\int_1^x t^2 dt =$$

$$\frac{d}{dx}\int_{\pi/6}^x \cos t\, dt =$$

Second Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\, dt =$$

$$\frac{d}{dx}\int_x^4 t^2 dt =$$

$$\frac{d}{dx}\int_x^a f(t)\, dt =$$

$$\frac{d}{dx}\int_{\pi/6}^{x^2} \cos t\, dt =$$

Second Fundamental Theorem of Calculus (Chain Rule Version):

$$\frac{d}{dx}\int_a^{g(x)} f(t)\, dt =$$

Ex. Use the Second Fundamental Theorem to evaluate:

(a) $\frac{d}{dx}\int_3^x \sqrt{1+t^2}\, dt =$

(b) $\frac{d}{dx}\int_2^x \tan(t^3)\, dt =$

(c) $\frac{d}{dx}\int_{-1}^{x^3} \frac{1}{1+t}\, dt =$

(d) $\frac{d}{dx}\int_2^{\sin x} \sqrt[3]{1+t^2}\, dt =$

Ex. The graph of a function f consists of a quarter circle and line segments. Let g be the function given by

$$g(x) = \int_0^x f(t)\,dt.$$

(a) Find $g(0), g(-1), g(2), g(5)$.

Graph of f

(b) Find all values of x on the open interval $(-1, 5)$ at which g has a relative maximum. Justify your answer.

(c) Find the absolute minimum value of g on $[-1, 5]$ and the value of x at which it occurs. Justify your answer.

(d) Find the x-coordinate of each point of inflection of the graph of g on $(-1, 5)$. Justify your answer.

CALCULUS
WORKSHEET ON SECOND FUNDAMENTAL THEOREM
AND FUNCTIONS DEFINED BY INTEGRALS

1. Find the derivatives of the functions defined by the following integrals:

(a) $\int_0^x \frac{\sin t}{t}\,dt$ (b) $\int_0^x e^{-t^2}\,dt$ (c) $\int_1^{\cos x} \frac{1}{t}\,dt$

(d) $\int_0^1 e^{\tan^2 t}\,dt$ (e) $\int_x^{x^2} \frac{1}{2t}\,dt, x>0$ (f) $\int_x^2 \cos\left(t^2\right)\,dt$

(g) $\int_1^{\sqrt{x}} \frac{s^2}{s^2+1}\,ds$ (h) $\int_{-5}^{\cos x} t\cos\left(t^3\right)\,dt$ (i) $\int_{\tan x}^{17} \sin\left(t^4\right)\,dt$

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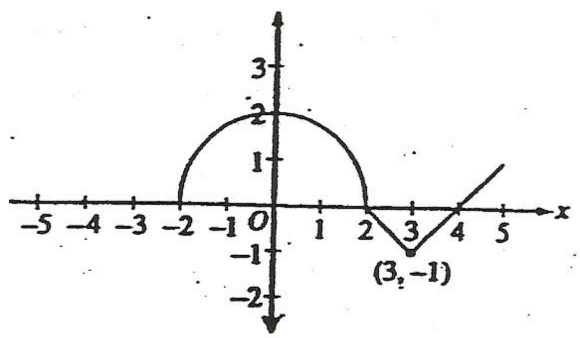
2. The graph of a function f consists of a semicircle and two line segments as shown. Let g be the function given by $g(x)=\int_0^x f(t)\,dt.$



(a) Find $g(0), g(3), g(-2),$ and $g(5)$.

(b) Find all values of x on the open interval $(-2,5)$ at which g has a relative maximum. Justify your answers.

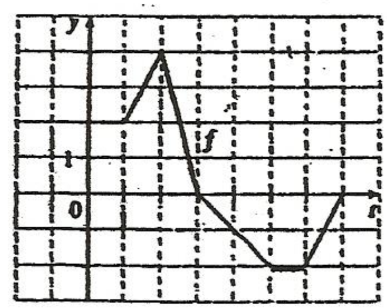
(c) Find the absolute minimum value of g on the closed interval $[-2,5]$ and the value of x at which it occurs. Justify your answer.

(d) Write an equation for the line tangent to the graph of g at $x = 3$.

(e) Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-2,5)$. Justify your answer.

(f) Find the range of g.

3. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown.



(a) Evaluate $g(0)$, $g(1)$, $g(2)$, and $g(6)$.

(b) On what intervals is g increasing?

(c) Where does g have a maximum value? What is the maximum value?

(d) Where does g have a minimum value? What is the minimum value?

(e) Sketch a rough graph of g on $[0, 7]$.

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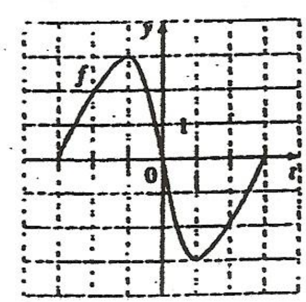
4. Let $g(x) = \int_{-3}^x f(t)\,dt$, where f is the function whose graph is shown.



(a) Evaluate $g(-3)$ and $g(3)$.

(b) At what values of x is g increasing? Justify.

(c) At what values of x does g have a maximum value? Justify.

(d) At what values of x does g have a minimum value? Justify.

(e) At what values of x does g have an inflection point? Justify.

5. Use the function f in the figure and the function g defined by

$$g(x) = \int_0^x f(t)\,dt.$$

(a) Complete the table.

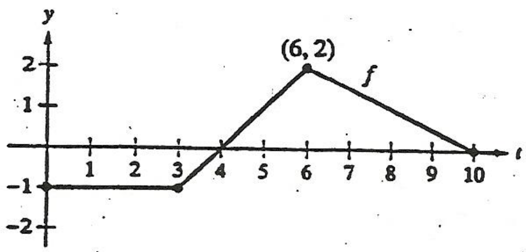
x	0	1	2	3	4	5	6	7	8	9	10
$g(x)$											

(b) Plot the points from the table in part (a).

(c) Where does g have its minimum?

(d) Which four consecutive points are collinear?

(e) Between which two consecutive points does g increase at the greatest rate?



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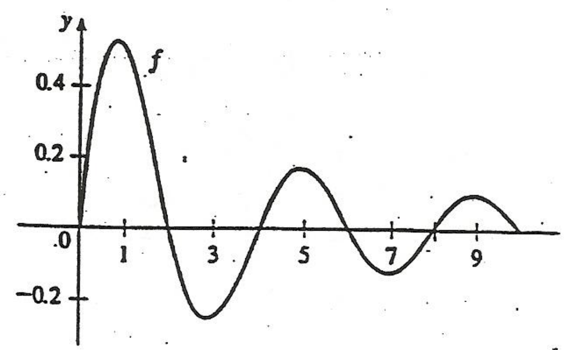
5. If $F(x) = \int_0^x f(t)\,dt$

a. Identify all critical numbers of F(x).

b. On what interval(s) is F(x) decreasing?

c. On what interval(s) is F(x) concave up?



CALCULUS
WORKSHEET 2 ON FUNCTIONS DEFINED BY INTEGRALS

1. Find the equation of the tangent line to the curve $y = F(x)$ where $F(x) = \int_1^x \sqrt[3]{t^2 + 7}\, dt$ at the point on the curve where $x = 1$.

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2. Suppose that $5x^3 + 40 = \int_c^x f(t)\, dt$.

(a) What is $f(x)$?

(b) Find the value of c.

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3. If $F(x) = \int_{-4}^x (t-1)^2 (t+3)\, dt$, for what values of x is F decreasing? Justify your answer.

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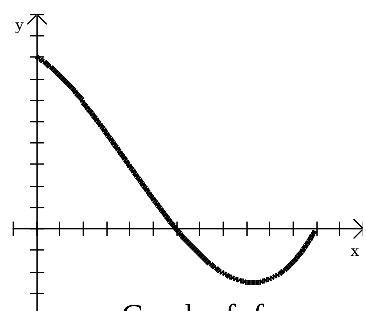
4. Let $H(x) = \int_0^x f(t)\, dt$ where f is the continuous function with domain [0, 12] shown on the right.



Graph of f

(a) Find $H(0)$.

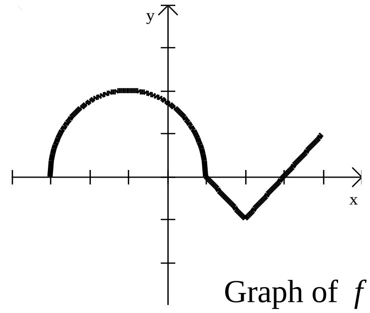
(b) On what interval(s) of x is H increasing? Justify your answer.

(c) On what interval(s) of x is H concave up? Justify your answer.

(d) Is $H(12)$ positive or negative? Explain.

(e) For what value of x does H achieve its maximum value? Explain.

5. The graph of a function f consists of a semicircle and two line segments as shown on the right.
Let $g(x) = \int_1^x f(t)\,dt$.



Graph of f

(a) Find $g(1)$, $g(3)$, $g(-1)$.

(b) On what interval(s) of x is g decreasing? Justify your answer.

(c) Find all values of x on the open interval $(-3, 4)$ at which g has a relative minimum. Justify your answer.

(d) Find the absolute maximum value of g on the interval $[-3, 4]$ and the value of x at which it occurs. Justify your answer.

(e) On what interval(s) of x is g concave up? Justify your answer.

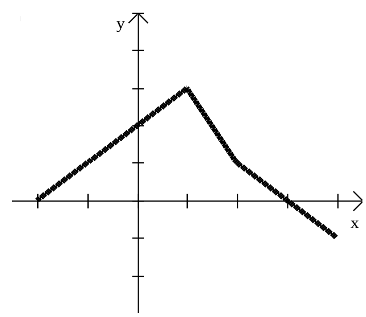
(f) For what value(s) of x does the graph of g have an inflection point? Justify your answer.

(g) Write an equation for the line tangent to the graph of g at $x = -1$.

6. The graph of the function f, consisting of three line segments, is shown on the right.
Let $g(x) = \int_1^x f(t)\,dt$.

(a) Find $g(2)$, $g(4)$, $g(-2)$.

(b) Find $g'(0)$ and $g'(3)$.



Graph of f

(c) Find the instantaneous rate of change of g with respect to x at $x = 2$.

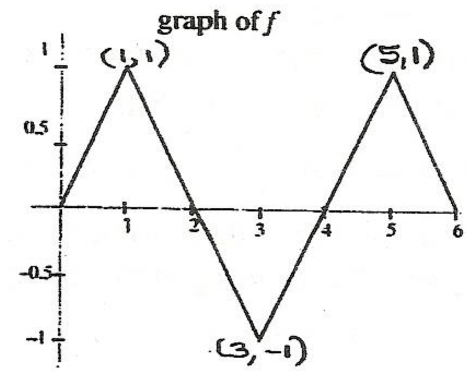
(d) Find the absolute maximum value of g on the interval $[-2, 4]$. Justify your answer.

(e) The second derivative of g is not defined at $x = 1$ and at $x = 2$. Which of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

CALCULUS
WORKSHEET 3 ON FUNCTIONS DEFINED BY INTEGRALS

Work the following on notebook paper.

1. The function g is defined on the interval [0, 6] by $g(x) = \int_0^x f(t)\,dt$ where f is the function graphed in the figure.



(a) For what values of x, $0 < x < 6$, does g have a relative maximum? Justify your answer.

(b) For what values of x is the graph of g concave down? Justify your answer.

(c) Write an equation for the tangent line to g at the point where $x = 3$.

(d) Sketch a graph of the function g. List the coordinates of all critical point and inflection points.

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2. Suppose that f' is a continuous function, that $f(1) = 13$, and that $f(10) = 7$. Find the average value of f' over the interval [1, 10].

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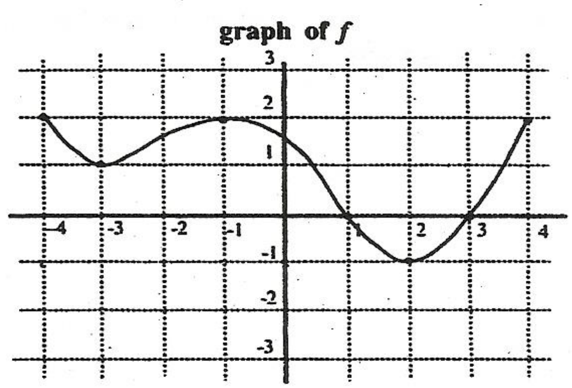
3. The graph of a differentiable function f on the closed interval $[-4, 4]$ is shown.

Let $G(x) = \int_{-4}^x f(t)\,dt$ for $-4 \le x \le 4$.



(a) Find $G(-4)$.

(b) Find $G'(-4)$.

(c) On which interval or intervals is the graph of G decreasing? Justify your answer.

(d) On which interval or intervals is the graph of G concave down? Justify your answer.

(e) For what values of x does G have an inflection point? Justify your answer.

4. The function F is defined for all x by $F(x) = \int_0^{x^2} \sqrt{t^2 + 8}\, dt$.

(a) Find $F'(x)$.

(b) Find $F'(1)$.

(c) Find $F''(x)$.

(d) Find $F''(1)$.

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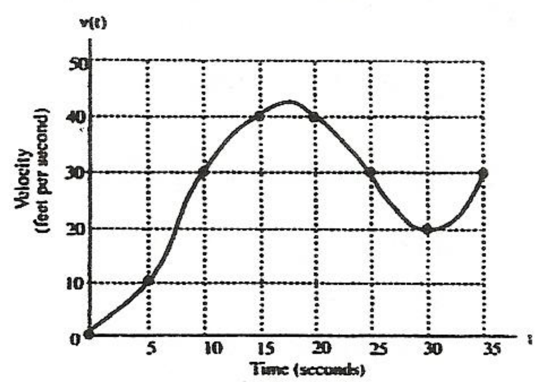
5. If $F(x) = \int_x^{-5} (t^2 - t - 6)\, dt$, on what intervals is F decreasing?



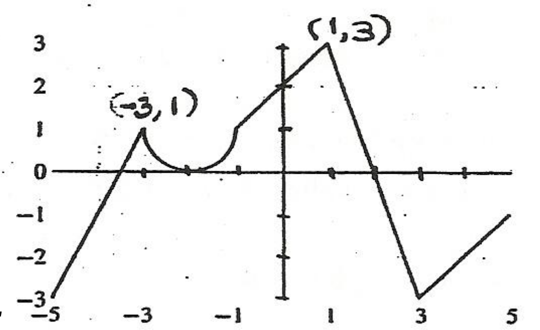
6. The graph of the velocity $v(t)$, in ft/sec, of a car traveling on a straight road, for $0 \le t \le 35$, is shown in the figure.

(a) Find the average acceleration of the car, in ft/sec^2, over the interval $0 \le t \le 35$.

(b) Find an approximation for the acceleration of the car, in ft/sec^2, at $t = 20$. Show your computations.

(c) Approximate $\int_5^{35} v(t)\, dt$ with a Riemann sum, using the midpoints of three subintervals of equal length. Explain the meaning of this integral.

7. The function F is defined for all x by $F(x) = \int_0^x f(t)\,dt$,



where f is the function graphed in the figure. The graph of f is made up of straight lines and a semicircle.

(a) For what values of x is F decreasing? Justify your answer.

(b) For what values of x does F have a local maximum? A local minimum? Justify your answer.

(c) Evaluate $F(2)$, $F'(2)$, and $F''(2)$.

(d) Write an equation of the line tangent to the graph of F at $x = 4$.

(e) For what values of x does F have an inflection point? Justify your answer.

Answers to Worksheets on Second Fund. Th. & Functions Defined by Integrals

1. (a) $\dfrac{\sin x}{x}$

   (b) e^{-x^2}

   (c) - tan x

   (d) 0

   (e) $\dfrac{1}{2x}$

   (f) $-\cos\ \ x^2$

   (g) $\dfrac{x}{2\sqrt{x}\ \ x+1}$

   (h) $-\sin x \cos x \cos\ \ \cos^3 x$

   (i) $-\sin\ \ \tan^4 x\ \ \sec^2 x$

2. (a) 0, $\pi - \dfrac{1}{2}$, $-\pi$, $\pi - \dfrac{1}{2}$

   (b) g has a rel. max. at x = 2 because $g'\ \ x\ = f\ \ x$ changes from positive to negative there.

   (c) Abs. min. = $-\pi$ at x = - 2 (Justify with Candidates' Test.)

   (d) $y - \left(\pi - \dfrac{1}{2}\right) = -\ \ x - 3$

   (e) g has an I.P at x = 0 because g' changes from increasing to decreasing there.
   g has an I.P at x = 3 because g' changes from decreasing to increasing there.

   (f) $[-\pi, \pi]$

3. (a) 0, 2, 5, 3

   (b) g is increasing on (0, 3) since g' is positive there.

   (c) Max. value = 7 at x = 3 (Justify with Candidates' Test.)

   (d) Min. value = 0 at x = 0 (Justify with Candidates' Test.)

   (e) $y - \dfrac{13}{2} = -\ \ x - 4$

4. (a) g is decreasing on $\left(1,\ 2\dfrac{1}{2}\right)$ and (4, 5) because $g'\ \ x\ = f\ \ x$ is negative there.

   (b) g has a rel. max. at x = 1 and at x = 4 because $g'\ \ x\ = f\ \ x$ changes from positive to negative there.

   (c) g is concave down on $\left(\dfrac{1}{2}, 1\dfrac{3}{4}\right)$ because $g'\ \ x\ = f\ \ x$ is decreasing there.

   (d) g has an I.P at $x = \dfrac{1}{2}$, $x = 1\dfrac{3}{4}$, and $x = 3\dfrac{1}{4}$ because g' changes from increasing to decreasing or vice versa there.

Worksheet 2 on Functions Defined by Integrals

1. $y = 2x - 2$

2. (a) $15x^2$ (b) -2

3. F is decreasing on $x < -3$ because $F' \;\; x \;\; < 0$ there.

4. (a) 0

(b) H is increasing on (0, 6) because $H' \;\; x \;\; = f \;\; x$ is positive there.

(c) H is concave up on (9.5, 12) because $H' \;\; x \;\; = f \;\; x$ is increasing there.

(d) $H \;\; 12$ is positive because there is more area above the x-axis than below.

(e) H achieves its maximum value at $x = 6$ because

$H \;\; 0 \;\; = 0$ and $H \;\; 6$ and $H \;\; 12$ are positive and $H \;\; 6 \;\; > H \;\; 12$.

5. (a) 0, -1, $-\pi$

(b) g is decreasing on (1, 3) because $g' \;\; x \;\; = f \;\; x$ is negative there.

(c) g has a relative minimum at $x = 3$ because $g' \;\; x \;\; = f \;\; x$ changes from negative to positive there.

(d) Abs. max. = 0 at $x = 1$ (Justify with Candidates' Test.)

(e) g is concave up on $-3, -1$ and $2, 4$ because $g' \;\; x \;\; = f \;\; x$ is increasing there.

(f) g has an inflection point at $x = -1$ and $x = 2$ because $g' \;\; x \;\; = f \;\; x$ changes from increasing to decreasing or vice versa there.

(g) $y + \pi = 2 \;\; x + 1$

6. (a) 2, 2, $-\dfrac{9}{2}$

(b) 2, 0

(c) 1

(d) Abs. max = $\dfrac{5}{2}$ at $x = 3$ (Justify with Candidates' Test.)

(e) g has an inflection point at $x = 1$ because $g' \;\; x \;\; = f \;\; x$ changes from increasing to decreasing there. g does not have an inflection point at $x = 2$ because $g' \;\; x \;\; = f \;\; x$ is decreasing for $1 < x < 2$ and continues to decrease on $2 < x < 4$.

Worksheet 3 on Functions Defined by Integrals

1. (a) g has a rel. max. at x = 2 because $g'\ \ x$, which is $f\ \ x$, changes from positive to negative there.

 (b) g is concave down on (1, 3) and (5, 6) because $g'\ \ x$, which is $f\ \ x$, is decreasing there.

 (c) $y-\frac{1}{2}=-\ \ x-3$ (d) graph

2. $-\frac{2}{3}$

3. (a) 0 (b) 2

 (c) G is decreasing on (1, 3) because $G'\ \ x$, which is $f\ \ x$, is negative there.

 (d) G has a rel. min. at x = 3 because $G'\ \ x$, which is $f\ \ x$, changes from negative to positive there.

 (e) G is concave down on $-4,-3$ and $-1,2$ because $G'\ \ x$, which is $f\ \ x$, is decreasing there.

 (f) G has an inflection point at $x=-3,\ x=-1,$ and $x=2$ because $G'\ \ x$, which is $f\ \ x$, changes from decreasing to increasing or vice versa there.

4. (a) $2x\sqrt{x^4+8}$ (b) 6

 (c) $\frac{4x^4}{\sqrt{x^4+8}}+2\sqrt{x^4+8}$ (d) $7\frac{1}{3}$

5. F is decreasing on $x<-2$ and $x>3$ because F' is negative there.

6. (a) $\frac{6}{7}\,\text{ft}/\text{sec}^2$

 (b) $-\ 2\ \text{ft}/\text{sec}^2$ (using (20, 40) and (25, 30) to estimate the slope)

 (c) (10)(30) + (10)(40) + (10)(20) = 900 ft.
 This integral represents the approximate distance in feet that the car has traveled from t = 5 seconds to t = 35 seconds.

7. (a) F is decreasing on $-5,-3.5$ and $2,5$ because $F'\ \ x$, which is $f\ \ x$, is negative there.

 (b) F has a local minimum at $x=-3.5$ because $F'\ \ x$, which is $f\ \ x$, changes from negative to positive there. F has a local maximum at x = 2 because $F'\ \ x$, which is $f\ \ x$, changes from positive to negative there.

 (c) 4, 0, -3

 (d) $y-0=-2\ \ x-4$

 (e) F has an inflection point at $x=-3,\ x=-2,\ x=1,$ and $x=3$ because $F'\ \ x$, which is $f\ \ x$, changes from increasing to decreasing or vice versa there.